# SMALL GAPS OF GOE 

RENJIE FENG, GANG TIAN AND DONGYI WEI


#### Abstract

In this article, we study the smallest gaps of the Gaussian orthogonal ensemble (GOE) with the joint density (5). The main result is that the smallest gaps, after normalized by $n$, will tend to a Poisson distribution, and the limiting density of the $k$-th normalized smallest gaps is $2 x^{2 k-1} e^{-x^{2}} /(k-1)!$.


## 1. Introduction

The problem regarding the spacings of eigenvalues is one of the most important problems in random matrix theory. The gap probability of eigenvalues for the classical random matrices GOE, GUE, GSE and its universality for more general ensembles such as the Wigner matrices are studied intensively and pretty wellknown [1, 7, 9, 11, 23, 29]. There are also results on the single spacing of eigenvalues for the classical matrices and some universal ensembles [11, 24, 28, 29. But there are only a few results regarding the extreme gaps. The motivations to study the extreme gaps of eigenvalues of random matrices come from many different areas such as conjectures regarding the extreme gaps for zeros of Riemann zeta function [10, 21, quantum chaos [4, 5] and quantum information theory [26. Now let's give a brief review of the existing results.

The way to derive the smallest gaps for the determinantal point processes is basically well established. The distributions of the smallest gaps of CUE and GUE were first obtained by Vinson using the moment method 30. In [27, Soshinikov investigated the smallest gaps for any determinantal point process on the real line with a translation invariant kernel and proved that some Poisson distribution can be observed in the limit. Then Ben Arous-Bourgade in [3] applied Soshinikov's method to derive the joint density of the smallest gaps of CUE and GUE, and they proved that the $k$-th smallest gaps of CUE and GUE, normalized by $n^{4 / 3}$, have the limiting density proportional to

$$
\begin{equation*}
x^{3 k-1} e^{-x^{3}} \tag{1}
\end{equation*}
$$

here, the joint density of GUE is

$$
\begin{equation*}
\frac{1}{Z_{n, 2}} e^{-n \sum_{i=1}^{n} \lambda_{i}^{2}} \prod_{1 \leq i<j \leq n}\left|\lambda_{i}-\lambda_{j}\right|^{2} \tag{2}
\end{equation*}
$$

where $Z_{n, 2}$ is the normalization constant. Later on, Figalli-Guionnet derived the smallest gaps for some invariant multimatrix Hermitian ensembles 17. As a remark, the determinantal structure is essential in the proofs in 3, 17, 27, 30,

[^0]In [15], we derived the smallest gaps of eigenangles of $\mathrm{C} \beta \mathrm{E}$ beyond the determinantal case for any positive integer $\beta$. For the two-dimensional point process

$$
\chi^{(n)}=\sum_{i=1}^{n} \delta_{\left(n^{\frac{\beta+2}{\beta+1}}\left(\theta_{i+1}-\theta_{i}\right), \theta_{i}\right)},
$$

we proved that $\chi^{(n)}$ tends to a Poisson point process $\chi$ as $n \rightarrow \infty$ with intensity

$$
\mathbb{E} \chi(A \times I)=\frac{A_{\beta}|I|}{2 \pi} \int_{A} u^{\beta} d u
$$

where $A \subset \mathbb{R}_{+}$is any bounded Borel set, $I \subset(-\pi, \pi)$ is an interval, $|I|$ is the Lebesgue measure of $I$ and

$$
\begin{equation*}
A_{\beta}=(2 \pi)^{-1} \frac{(\beta / 2)^{\beta}(\Gamma(\beta / 2+1))^{3}}{\Gamma(3 \beta / 2+1) \Gamma(\beta+1)} \tag{3}
\end{equation*}
$$

In particular, the result holds for COE, CUE and CSE with

$$
A_{1}=\frac{1}{24}, A_{2}=\frac{1}{24 \pi}, \quad A_{4}=\frac{1}{270 \pi}
$$

correspondingly.
As a direct consequence, let's denote $t_{k, \beta}^{n}$ as the $k$-th smallest gap of $\mathrm{C} \beta \mathrm{E}$ where $t_{1, \beta}^{n}<t_{2, \beta}^{n}<t_{3, \beta}^{n} \cdots$ and define

$$
\tau_{k, \beta}^{n}=n^{(\beta+2) /(\beta+1)} \times\left(A_{\beta} /(\beta+1)\right)^{1 /(\beta+1)} t_{k, \beta}^{n}
$$

then we have the limiting density

$$
\begin{equation*}
\lim _{n \rightarrow+\infty} \mathbb{P}\left(\tau_{k, \beta}^{n} \in A\right)=\int_{A} \frac{\beta+1}{(k-1)!} x^{k(\beta+1)-1} e^{-x^{\beta+1}} d x \tag{4}
\end{equation*}
$$

For general $\mathrm{C} \beta \mathrm{E}$, there is no determinantal structure as CUE and the whole proof in [15] is based on the Selberg integral.

The decay order $\sqrt{32 \log n} / n$ of the largest gaps of CUE and GUE was predicted by Vinson in [30, and the proof is given by Ben Arous-Bourgade in [3]. The same decay order for the largest gaps of some invariant multimatrix Hermitian matrices is derived by Figalli-Guionnet in [17. Recently, the fluctuations of the largest gaps of CUE and GUE are further derived in [16].

But there is no previous result on the extreme gaps for GOE. There are some essential difficulties for GOE compared with GUE. For GUE, it is a determinantal point process so that one can express the point correlation functions explicitly and apply the Hadamard-Fisher inequality to control the estimates. This is not the case for GOE even though GOE has a Pfaffian structure. One can only express the point correlation functions of GOE as integrals of the joint density. This causes many difficulties and all the proofs require delicate estimates of the integrals. In this paper, we will derive the smallest gaps of GOE and this is the first result regarding the extreme gaps of GOE. Our arguments follow the approach we developed in 15 .

For GOE, the joint density of the eigenvalues is

$$
\begin{equation*}
\frac{1}{G_{n}} e^{-\sum_{i=1}^{n} \lambda_{i}^{2} / 2} \prod_{1 \leq i<j \leq n}\left|\lambda_{i}-\lambda_{j}\right| \tag{5}
\end{equation*}
$$

with respect to the Lebesgue measure on $\mathbb{R}^{n}$. Here, the normalization constant

$$
\begin{equation*}
G_{n}:=\int_{\mathbb{R}^{n}} d \lambda_{1} \cdots d \lambda_{n} e^{-\sum_{i=1}^{n} \lambda_{i}^{2} / 2} \prod_{i<j}\left|\lambda_{i}-\lambda_{j}\right| \tag{6}
\end{equation*}
$$

is (Proposition 4.7.1 in [18])

$$
\begin{equation*}
G_{n}=(2 \pi)^{n / 2} \prod_{j=0}^{n-1} \frac{\Gamma(1+(j+1) / 2)}{\Gamma(3 / 2)} \tag{7}
\end{equation*}
$$

In fact, one may view the above joint density as the one-component log-gas of $n$ particles with charge $q=1$ on the real line and the Hamiltonian is

$$
H\left(\lambda_{1}, \cdots, \lambda_{n}\right)=\sum_{i=1}^{n} \lambda_{i}^{2} / 2-\sum_{i<j} \log \left|\lambda_{i}-\lambda_{j}\right|
$$

Now let's consider the following point process on $\mathbb{R}_{+}$

$$
\begin{equation*}
\chi^{(n)}=\sum_{i=1}^{n-1} \delta_{n\left(\lambda_{(i+1)}-\lambda_{(i)}\right)} \tag{8}
\end{equation*}
$$

where $\lambda_{(i)}(1 \leq i \leq n)$ is the increasing rearrangement of $\lambda_{i}(1 \leq i \leq n)$. The main result of this article is

Theorem 1. Let $\lambda_{1}, \cdots, \lambda_{n}$ be eigenvalues of GOE, then the point process $\chi^{(n)}$ will converge to a Poisson point process $\chi$ as $n \rightarrow+\infty$ with intensity

$$
\mathbb{E} \chi(A)=\frac{1}{4} \int_{A} u d u
$$

where $A \subset \mathbb{R}_{+}$is any bounded Borel set.
As a direct consequence of Theorem 1 we will have
Corollary 1. Let's denote $t_{k}^{n}$ as the $k$-th smallest gap and $\tau_{k}^{n}=2^{-3 / 2} n t_{k}^{n}$, then

$$
\begin{equation*}
\lim _{n \rightarrow+\infty} \mathbb{P}\left(\tau_{k}^{n} \in A\right)=\int_{A} \frac{2}{(k-1)!} x^{2 k-1} e^{-x^{2}} d x \tag{9}
\end{equation*}
$$

for any bounded interval $A \subset \mathbb{R}_{+}$.
As a remark, the factor $1 / 4$ in Theorem 1 is quite meaningful. In fact, the main observation in Lemma is that

$$
\begin{equation*}
1 / 4=\left(G_{n-2 k, k} / G_{n}\right)^{1 / k} \tag{10}
\end{equation*}
$$

i.e., its $k$-th power is the quotient of the generalized partition function of the twocomponent log-gas (where the system consists of $n-2 k$ particles with charge $q=1$ and $k$ particles with charge $q=2$ ) and the partition function of the one-component log-gas (see $\$ 2$ for these definitions). Actually, one of the crucial ideas in the whole proof is that one can bound the integrals of the joint density of one-component loggas by the generalized partition functions of two-component log-gas (see Lemma 11 in (6).
1.1. Remarks. One may consider the smallest gaps for $\mathrm{G} \beta \mathrm{E}$ with the joint density

$$
\begin{equation*}
\frac{1}{Z_{n, \beta}} e^{-n \beta \sum_{i=1}^{n} \lambda_{i}^{2} / 2} \prod_{1 \leq i<j \leq n}\left|\lambda_{i}-\lambda_{j}\right|^{\beta} \tag{11}
\end{equation*}
$$

where $\beta>0$ and $Z_{n, \beta}$ is the normalization constant. Note that compared with (5), the joint density (11) with $\beta=1$ has a factor $n$ in the exponential function, this will cause an extra factor $\sqrt{n}$ for the spacings of eigenvalues, i.e., the smallest gap is of order $n^{-3 / 2}$ under the joint density (11) with $\beta=1$ for GOE.

By comparing the limiting densities (11) (9) with (4) with $\beta=1,2$, it is believed that the smallest gaps of $\mathrm{G} \beta \mathrm{E}$ have the same limiting behaviors as $\mathrm{C} \beta \mathrm{E}$ and we propose the following conjecture.
Conjecture 1. Let's denote $t_{k, \beta}^{n}$ as the $k$-th smallest gap of $G \beta E$ with the joint density (11), then there exists some constant $c_{\beta}$ depending on $\beta$ such that

$$
\begin{equation*}
\tau_{k, \beta}^{n}=c_{\beta} n^{(\beta+2) /(\beta+1)} t_{k, \beta}^{n} \tag{12}
\end{equation*}
$$

has the limiting density

$$
\begin{equation*}
\frac{\beta+1}{(k-1)!} x^{k(\beta+1)-1} e^{-x^{\beta+1}} \tag{13}
\end{equation*}
$$

as $n \rightarrow \infty$.
It seems that our strategy to prove the smallest gaps for GOE can be used to prove that of $\mathrm{G} \beta \mathrm{E}$ and more general ensembles with the joint density

$$
\begin{equation*}
\frac{1}{Z_{n, \beta, V}} e^{-n \beta \sum_{i=1}^{n} V\left(\lambda_{i}\right)} \prod_{1 \leq i<j \leq n}\left|\lambda_{i}-\lambda_{j}\right|^{\beta} \tag{14}
\end{equation*}
$$

It's very likely that Conjecture 1 is still true for general potential $V(x)$ with mild assumptions instead of $x^{2} / 2$. One of the difficulties is to prove some identity as (10) or some asymptotic limit as in Lemma 4 in [15]. Actually, there are some results only in the case of $\beta=2$, for example, Vinson derived the smallest gaps when the potential $V(x)$ is a real analytic potential which is regular and whose equilibrium measure supported on a single interval [30; while in [17], Figalli-Guionnet derived the universal results for the smallest gaps for some invariant multimatrix Hermitian matrices.

Recently, in 66, 22, Bourgade and Landon-Lopatto-Marcinek proved the universality for the extreme gaps in the bulk of the general Hermitian and symmetric Wigner matrices with assumptions.

In the end, let's mention some conjectures and results regarding the local statistics of many other important point processes that are related to the classical random matrix models. The local statistics of eigenvalues of the Laplacian of several integrable systems are believed to follow Poisson statistics [2], while for generic chaotic systems, such as non-arithmetic surfaces of negative curvature, they are expected to follow the GOE [5] (see [4] for the results about the smallest gaps between the first $N$ eigenvalues of the Laplacian on a rectangular billiard as $N$ large enough). In number theory, the local statistics of zeros of Riemann zeta function are expected to follow the GUE [10, 21]. In high energy physics, the numerical results in [19, 20] indicate that the local behaviors of the SYK model, which describes $n$ (an even integer) random interacting Majorana modes on a quantum dot [8, are similar to $\operatorname{GOE}(n=0 \bmod 8), \operatorname{GUE}(n=2,6 \bmod 8)$ and $\operatorname{GSE}(n=4 \bmod 8)$, i.e., the single

SYK model encodes the three classical random matrix models. We also refer to [12, 13, 14 for the mathematical results on the SYK model.

The organization of this article is as follows. In Section 2, we review some basic facts about the joint density of GOE, two-component log-gas, the Hermite polynomials and the Pfaffian of an antisymmetric matrix. In Section 3, we prove an important identity for the generalized partition functions of the two-component loggas of GOE. Its proof uses certain properties of Pfaffians and Hermite polynomials regarding GOE. In Section 4, we introduce and discuss two more auxiliary point processes. In Section 5, we prove the non-existence of successive small gaps. In Section 6, we establish certain integral inequalities for the two-component log-gas. In Section 7, we complete the proof of Theorem 1

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## 2. Preliminaries

In this section, we will first review some results regarding the joint density of GOE, two-component log-gas and the Hermite polynomials. Then we will recall the definition and several basic properties of the Pfaffian of an antisymmetric matrix.

As explained in [23] (see (5.2.9) and (6.1.2)-(6.1.5) in 23]), we can rewrite the joint density (5) as

$$
\left|J_{n}\left(x_{1}, \cdots, x_{n}\right)\right| / G_{n}
$$

where $J_{n}\left(x_{1}, \cdots, x_{n}\right)$ can be expressed in terms of a determinant as

$$
\begin{equation*}
J_{n}\left(x_{1}, \cdots, x_{n}\right):=e^{-\sum_{i=1}^{n} x_{i}^{2} / 2} \prod_{j<i}\left(x_{i}-x_{j}\right)=c_{n} \operatorname{det}\left[\varphi_{i-1}\left(x_{j}\right)\right]_{i, j=1, \cdots, n} \tag{15}
\end{equation*}
$$

and the partition function of the integration constant is

$$
\begin{align*}
G_{n} & =\int_{\mathbb{R}^{n}} d x_{1} \cdots d x_{n}\left|J_{n}\left(x_{1}, \cdots, x_{n}\right)\right|  \tag{16}\\
& =n!c_{n} \int_{x_{1}<\cdots<x_{n}} d x_{1} \cdots d x_{n} \operatorname{det}\left[\varphi_{i-1}\left(x_{j}\right)\right]_{i, j=1, \cdots, n},
\end{align*}
$$

where $c_{n}>0$ is a constant depending only on $n$ and

$$
\begin{equation*}
\varphi_{j}(x)=\left(2^{j} j!\sqrt{\pi}\right)^{-1 / 2} e^{-x^{2} / 2} H_{j}(x)=\left(2^{j} j!\sqrt{\pi}\right)^{-1 / 2} e^{x^{2} / 2}(-d / d x)^{j} e^{-x^{2}} \tag{17}
\end{equation*}
$$

are the "oscillator wave functions" orthogonal over $\mathbb{R}$ such that

$$
\int_{\mathbb{R}} \varphi_{j}(x) \varphi_{k}(x) d x=\delta_{j k}= \begin{cases}1, & \text { if } j=k  \tag{18}\\ 0, & \text { otherwise }\end{cases}
$$

Here, $\left\{H_{j}(x)\right\}$ are Hermite polynomials. From the following recurrence relations of Hermite polynomials

$$
\begin{equation*}
H_{j+1}(x)=2 x H_{j}(x)-2 j H_{j-1}(x), H_{j}^{\prime}(x)=2 j H_{j-1}(x) \tag{19}
\end{equation*}
$$

one deduces

$$
\begin{equation*}
\sqrt{2} \varphi_{j}^{\prime}(x)=\sqrt{j} \varphi_{j-1}(x)-\sqrt{j+1} \varphi_{j+1}(x), \quad j \geq 0 \tag{20}
\end{equation*}
$$

where we denote $\varphi_{-1}(x)=0$. Moreover, we have (see (5.47) in [18])

$$
\begin{equation*}
H_{j}(x)=\sum_{m=0}^{[j / 2]}(-1)^{m} 2^{j-m}\binom{j}{2 m} \frac{(2 m)!}{2^{m} m!} x^{j-2 m} \tag{21}
\end{equation*}
$$

and $H_{n}(x)$ is uniquely determined by the first equation of (19) and the initial condition $H_{0}(x)=1, H_{1}(x)=2 x$. From the expression of $H_{j}(x)$, we also have

$$
\begin{align*}
& \operatorname{span}\left\{x^{j} ; j \in \mathbb{Z} \cap[0, n]\right\}=\operatorname{span}\left\{H_{j}(x) ; j \in \mathbb{Z} \cap[0, n]\right\}  \tag{22}\\
& V_{n}:=\operatorname{span}\left\{x^{j} e^{-x^{2} / 2} ; j \in \mathbb{Z} \cap[0, n]\right\}=\operatorname{span}\left\{\varphi_{j}(x) ; j \in \mathbb{Z} \cap[0, n]\right\} \tag{23}
\end{align*}
$$

Actually, the joint density (5) can be identified with the Boltzmann factor of a particular one-component log-gas (see $\S 1.4$ in [18]). One can also define the two-component log-gas for the system that consists of $n_{1}$ particles with charge $q=1$ and $n_{2}$ particles with charge $q=2$. The two-component log-gas provides an interpolation between $\operatorname{GOE}(\beta=1)$ and $\operatorname{GSE}(\beta=4)$ (see [25] and $\S 6.7$ in [18]). For the two-component log-gas, the generalized partition function of the integration constant is

$$
\begin{equation*}
G_{n_{1}, n_{2}}:=\int_{\mathbb{R}^{n_{1}+n_{2}}} d \lambda_{1} \cdots d \lambda_{n_{1}+n_{2}} e^{-\sum_{i=1}^{n_{1}+n_{2}} q_{i} \lambda_{i}^{2} / 2} \prod_{j<k}\left|\lambda_{j}-\lambda_{k}\right|^{q_{j} q_{k}} \tag{24}
\end{equation*}
$$

where $q_{j}=1$ for $1 \leq j \leq n_{1}$ and $q_{j}=2$ for $n_{1}+1 \leq j \leq n_{1}+n_{2}$.
Let

$$
J_{n_{1}, n_{2}}\left(x_{1}, \cdots, x_{n_{1}+n_{2}}\right):=e^{-\sum_{i=1}^{n_{1}+n_{2}} q_{i} x_{i}^{2} / 2} \prod_{j<i}\left(x_{i}-x_{j}\right)^{q_{i} q_{j}}
$$

where $q_{j}=1$ for $1 \leq j \leq n_{1}$ and $q_{j}=2$ for $n_{1}+1 \leq j \leq n_{1}+n_{2}$. Then we have

$$
J_{n_{1}, n_{2}}\left(x_{1}, \cdots, x_{n_{1}+n_{2}}\right)=\prod_{j=1}^{n_{2}} \frac{\partial}{\partial y_{n_{1}+2 j}} J_{n_{1}+2 n_{2}}\left(y_{1}, \cdots, y_{n_{1}+2 n_{2}}\right)
$$

where the right hand side is evaluated at $y_{j}=x_{j}$ for $j \in \mathbb{Z} \cap\left[1, n_{1}\right]$ and $y_{n_{1}+2 j}=$ $y_{n_{1}+2 j-1}=x_{n_{1}+j}$ for $j \in \mathbb{Z} \cap\left[1, n_{2}\right]$. Therefore, differentiating (15), we have
and

Here, $\Delta_{j}=\left\{x_{1}<\cdots<x_{j}\right\} \subset \mathbb{R}^{j}$ is a simplex. We also have

$$
\begin{equation*}
G_{n}=G_{n, 0} \tag{26}
\end{equation*}
$$

Now let's recall the definition of the Pfaffian of an antisymmetric matrix of even size (see Definition 6.1.4 in [18]): Let $X=\left[\alpha_{i j}\right]_{i, j=1, \cdots, 2 N}$ be an antisymmetric matrix. Then the Pfaffian of $X$ is defined by

$$
\begin{align*}
\operatorname{Pf} X & =\sum_{P(2 l)>P(2 l-1)}^{*} \varepsilon(P) \prod_{l=1}^{N} \alpha_{P(2 l-1) P(2 l)}  \tag{27}\\
& =\frac{1}{2^{N} N!} \sum_{P \in S_{2 N}} \varepsilon(P) \prod_{l=1}^{N} \alpha_{P(2 l-1) P(2 l)},
\end{align*}
$$

where in the first summation the $*$ denotes that the sum is restricted to distinct terms only and $\varepsilon(P)$ is the signature of the permutation $P$.

When $X$ is a $2 N \times 2 N$ antisymmetric matrix and $B$ is a general $2 N \times 2 N$ matrix, then we have (see (6.12) and (6.35) in [18])

$$
\begin{equation*}
(\operatorname{Pf} X)^{2}=\operatorname{det} X, \operatorname{Pf}\left(B^{T} X B\right)=(\operatorname{det} B)(\operatorname{Pf} X), \operatorname{Pf}(\lambda X)=\lambda^{N} \operatorname{Pf} X \tag{28}
\end{equation*}
$$

Here, the third identity follows from the definition (27).

## 3. Partition functions of two-component log-Gas

In this section, we will prove Lemma for the two-component log-gas of GOE. The proof is based on the properties of Pfaffians and Hermite polynomials regarding GOE (see [9, 18, 23, 25] for more details) and some integration techniques from Chapter 6 of 18 .

Lemma 1. For any positive integers $n, k, n \geq 2 k$, we have $G_{n-2 k, k}=2^{-2 k} G_{n}$.
The following Lemma 2 and Lemma 3 give the expressions of $G_{n_{1}, n_{2}}$ for the cases $n_{1}$ even and $n_{1}$ odd separately, where one can express the generalized partition functions $G_{n_{1}, n_{2}}$ in terms of Pfaffians via the method of integration over alternate variables (see §6.3.2 in [18]).
Lemma 2. For the case $n_{1}$ even, we have

$$
G_{n_{1}, n_{2}}=\left(n_{1}!n_{2}!\right) c_{n_{1}+2 n_{2}}\left[\zeta^{n_{1} / 2}\right] \operatorname{Pf}\left[\beta_{j, k}+\zeta \alpha_{j, k}\right]_{j, k=1, \cdots, n_{1}+2 n_{2}}
$$

where $\left[\zeta^{j}\right] f$ denotes the coefficient of $\zeta^{j}$ in the power series expansion of $f$ and

$$
\begin{aligned}
\alpha_{j, k} & :=\int_{\mathbb{R}^{2}} \varphi_{k-1}(x) \varphi_{j-1}(y) \operatorname{sgn}(x-y) d x d y \\
\beta_{j, k} & :=\int_{\mathbb{R}}\left(\varphi_{k-1}^{\prime}(x) \varphi_{j-1}(x)-\varphi_{k-1}(x) \varphi_{j-1}^{\prime}(x)\right) d x
\end{aligned}
$$

Proof. According to (25), as in the proof of Proposition 6.3.4 in [18, applying the method of integration over alternate variables to integrate over $x_{1}, x_{3}, \cdots, x_{n_{1}-1}$, and expanding the resulting determinant to integrate over all the rest variables gives

$$
G_{n_{1}, n_{2}}=\frac{\left(n_{1}!\right) c_{n_{1}+2 n_{2}}}{\left(n_{1} / 2\right)!} \sum_{P \in S_{n_{1}+2 n_{2}}} \varepsilon(P) \prod_{l=1}^{n_{1} / 2} a_{P(2 l-1), P(2 l)} \prod_{l=n_{1} / 2+1}^{n_{1} / 2+n_{2}} b_{P(2 l-1), P(2 l)}
$$

where

$$
a_{j, k}:=\int_{\mathbb{R}} d x \varphi_{k-1}(x) \int_{-\infty}^{x} d y \varphi_{j-1}(y)
$$

$$
b_{j, k}:=\int_{\mathbb{R}} \varphi_{k-1}^{\prime}(x) \varphi_{j-1}(x) d x
$$

Making the restriction $P(2 l)>P(2 l-1)$, we further have

$$
G_{n_{1}, n_{2}}=\frac{\left(n_{1}!\right) c_{n_{1}+2 n_{2}}}{\left(n_{1} / 2\right)!} \sum_{P(2 l)>P(2 l-1)} \varepsilon(P) \prod_{l=1}^{n_{1} / 2} \alpha_{P(2 l-1), P(2 l)} \prod_{l=n_{1} / 2+1}^{n_{1} / 2+n_{2}} \beta_{P(2 l-1), P(2 l)}
$$

Then the result is a consequence of the definition of a Pfaffian.
Lemma 3. For the case $n_{1}$ odd, let $n=n_{1}+2 n_{2}$, then we have

$$
\frac{G_{n_{1}, n_{2}}}{\left(n_{1}!n_{2}!\right) c_{n}}=\left[\zeta^{\left(n_{1}-1\right) / 2}\right] \operatorname{Pf}\left[\begin{array}{ll}
{\left[\beta_{j, k}+\zeta \alpha_{j, k}\right]_{j, k=1, \cdots, n}} & {\left[\nu_{j}\right]_{j=1, \cdots, n}} \\
-\left[\nu_{k}\right]_{k=1, \cdots, n} & 0
\end{array}\right]
$$

where $\alpha_{j, k}, \beta_{j, k}$ are defined in Lemma 2 and

$$
\nu_{k}:=\int_{\mathbb{R}} \varphi_{k-1}(x) d x
$$

Proof. With the same definitions of $a_{j, k}$ and $b_{j, k}$ as in the proof of Lemma 2 we apply the method of integration over alternate variables again to integrate over $x_{1}, x_{3}, \cdots, x_{n_{1}}$ first, then we expand the resulting determinant and integrate over all the rest variables to get

$$
\begin{aligned}
G_{n_{1}, n_{2}}= & \frac{\left(n_{1}!\right) c_{n_{1}+2 n_{2}}}{\left(\left(n_{1}-1\right) / 2\right)!} \sum_{P \in S_{n_{1}+2 n_{2}}} \varepsilon(P) \nu_{P\left(n_{1}\right)} \times \\
& \prod_{l=1}^{\left(n_{1}-1\right) / 2} a_{P(2 l-1), P(2 l)} \prod_{l=1}^{n_{2}} b_{P\left(n_{1}+2 l-1\right), P\left(n_{1}+2 l\right)} \\
= & \frac{\left(n_{1}!\right) c_{n_{1}+2 n_{2}}}{\left(\left(n_{1}-1\right) / 2\right)!} \sum_{P(2 l)>P(2 l-1) ; l=1, \cdots,\left(n_{1}-1\right) / 2+n_{2}} \varepsilon(P) \nu_{P\left(n_{1}+2 n_{2}\right)} \times \\
& \prod_{l=1}^{\left(n_{1}-1\right) / 2} \alpha_{P(2 l-1), P(2 l)} \prod_{l=\left(n_{1}-1\right) / 2}^{\left(n_{1}-1\right) / 2+n_{2}} \beta_{P(2 l-1), P(2 l)} .
\end{aligned}
$$

Here, we changed the order $P\left(n_{1}\right), P\left(n_{1}+1\right), \cdots, P\left(n_{1}+2 n_{2}\right) \rightarrow P\left(n_{1}+2 n_{2}\right), P\left(n_{1}\right)$, $\cdots, P\left(n_{1}+2 n_{2}-1\right)$, and made the restriction $P(2 l)>P(2 l-1)$. Now we write $\nu_{P\left(n_{1}+2 n_{2}\right)}=\nu_{P(n)}:=\nu_{P(n), n+1}=-\nu_{n+1, P(n)}$ in the above expression, then the result is again a consequence of the definition of a Pfaffian.

Now we need several properties of $\alpha_{j, k}, \beta_{j, k}$ and $\nu_{k}$. By (18) and (20), we first have

$$
\begin{equation*}
\beta_{k, k+1}=-\beta_{k+1, k}=\sqrt{2 k}, \beta_{j, k}=0 \text { for }|j-k| \neq 1 \tag{29}
\end{equation*}
$$

We also have the following
Lemma 4. Let $\alpha_{j, k}, \beta_{j, k}$ be defined in Lemma 图, $\nu_{k}$ be defined in Lemma 3, and let's define $\alpha_{0, k}=\alpha_{j, 0}=\nu_{0}=0$. Then we have
(a) for positive integers $j, k$, we have

$$
\sqrt{j-1} \alpha_{j-1, k}-\sqrt{j} \alpha_{j+1, k}=2 \sqrt{2} \delta_{j k}, \sqrt{j-1} \nu_{j-1}-\sqrt{j} \nu_{j+1}=0
$$

(b) $\nu_{j}=0$ for $j$ even; $\nu_{j}>0$ for $j$ odd.
(c) $\alpha_{j, k}=\alpha_{1, k} \nu_{j} / \nu_{1}$ for $0<j \leq k$.
(d) If $k$ is odd, then $\alpha_{j, k}=0$ for $0<j \leq k$; if $k$ is even $(k>0)$, then $\alpha_{1, k}>0$.
(e) If $n$ is even, $n>0, j, k \in \mathbb{Z} \cap[1, n]$, then

$$
\sum_{l=1}^{n} \beta_{j, l} \alpha_{l, k}=-4 \delta_{j k}
$$

Proof. Let's define a skew symmetric inner product $\langle\cdot \mid \cdot\rangle_{1}$ by

$$
\langle f \mid g\rangle_{1}:=\int_{\mathbb{R}^{2}} g(x) f(y) \operatorname{sgn}(x-y) d x d y
$$

then we have

$$
\alpha_{j, k}=\left\langle\varphi_{j-1} \mid \varphi_{k-1}\right\rangle_{1}=-\alpha_{k, j}
$$

Thanks to (18) and $\lim _{x \rightarrow \pm \infty} \varphi_{j}(x)=0$, we have

$$
\begin{aligned}
\left\langle\varphi_{j}^{\prime} \mid \varphi_{k}\right\rangle_{1} & =\int_{\mathbb{R}} d x \varphi_{k}(x)\left(\int_{-\infty}^{x} \varphi_{j}^{\prime}(y) d y-\int_{x}^{+\infty} \varphi_{j}^{\prime}(y) d y\right) \\
& =\int_{\mathbb{R}} d x \varphi_{k}(x)\left(2 \varphi_{j}(x)\right)=2 \delta_{j k}
\end{aligned}
$$

Hence, by (20), we will have

$$
\begin{aligned}
2 \sqrt{2} \delta_{j k} & =\left\langle\sqrt{2} \varphi_{j}^{\prime} \mid \varphi_{k}\right\rangle_{1}=\sqrt{j}\left\langle\varphi_{j-1} \mid \varphi_{k}\right\rangle_{1}-\sqrt{j+1}\left\langle\varphi_{j+1} \mid \varphi_{k}\right\rangle_{1} \\
& =\sqrt{j} \alpha_{j, k+1}-\sqrt{j+1} \alpha_{j+2, k+1}
\end{aligned}
$$

and thus we conclude the first identity of (a).
Similarly, we have
$0=\int_{\mathbb{R}} \sqrt{2} \varphi_{j}^{\prime}(x) d x=\int_{\mathbb{R}}\left(\sqrt{j} \varphi_{j-1}(x)-\sqrt{j+1} \varphi_{j+1}(x)\right) d x=\sqrt{j} \nu_{j}-\sqrt{j+1} \nu_{j+2}$,
which implies the second identity of (a).
If $j$ is even, we have

$$
\nu_{j}=\nu_{0} \prod_{l=0}^{(j-2) / 2} \frac{\sqrt{2 l}}{\sqrt{2 l+1}}=0
$$

By (17), we have $\varphi_{0}(x)>0$, and thus

$$
\nu_{1}=\int_{\mathbb{R}} \varphi_{0}(x) d x>0
$$

therefore, for $j$ odd, we have

$$
\nu_{j}=\nu_{1} \prod_{l=1}^{(j-1) / 2} \frac{\sqrt{2 l-1}}{\sqrt{2 l}}>0
$$

This shows that (b) is true.
By (a) where $\sqrt{j-1} \alpha_{j-1, k}-\sqrt{j} \alpha_{j+1, k}=0$ for $0<j<k$, we will have

$$
\begin{aligned}
& \alpha_{j, k}=\alpha_{0, k} \prod_{l=0}^{(j-2) / 2} \frac{\sqrt{2 l}}{\sqrt{2 l+1}}=0=\alpha_{1, k} \nu_{j} / \nu_{1} \text { for } j \text { even, } 0<j \leq k, \\
& \alpha_{j, k}=\alpha_{1, k} \prod_{l=1}^{(j-1) / 2} \frac{\sqrt{2 l-1}}{\sqrt{2 l}}=\alpha_{1, k} \nu_{j} / \nu_{1} \text { for } j \text { odd, } 0<j \leq k,
\end{aligned}
$$

and thus (c) is true.
Since $\alpha_{j, k}=-\alpha_{k, j}$, we have $\alpha_{k, k}=0$. If $k$ is odd, then $0=\alpha_{k, k}=\alpha_{1, k} \nu_{k} / \nu_{1}$ and $\nu_{1}>0, \nu_{k}>0$, then we must have $\alpha_{1, k}=0$ and $\alpha_{j, k}=\alpha_{1, k} \nu_{j} / \nu_{1}=0$ for $0<j \leq k$. If $k$ is even $(k>0)$, then $k \pm 1$ are odd and thus $\alpha_{k, k+1}=0=-\alpha_{k+1, k}, \nu_{k-1}>0$. By (a), we have

$$
\sqrt{k-1} \alpha_{k-1, k}=\sqrt{k-1} \alpha_{k-1, k}-\sqrt{k} \alpha_{k+1, k}=2 \sqrt{2} \delta_{k k}=2 \sqrt{2}
$$

and $\alpha_{1, k} \nu_{k-1} / \nu_{1}=\alpha_{k-1, k}>0$. Thus we must have $\alpha_{1, k}>0$, which completes (d).
Now we assume that $n$ is even, $n>0, j, k \in \mathbb{Z} \cap[1, n]$, then $n+1>k$ and $n+1$ is odd. By (d), we have $\alpha_{n+1, k}=-\alpha_{k, n+1}=0$. Thus by (29) and (a), we have

$$
\begin{aligned}
\sum_{l=1}^{n} \beta_{j, l} \alpha_{l, k} & =\sum_{l=1}^{n+1} \beta_{j, l} \alpha_{l, k}=-\sqrt{2(j-1)} \alpha_{j-1, k}+\sqrt{2 j} \alpha_{j+1, k} \\
& =(-\sqrt{2}) \cdot 2 \sqrt{2} \delta_{j k}=-4 \delta_{j k}
\end{aligned}
$$

which is (e).
For the evaluation of Pfaffians, we need the following abstract result.
Lemma 5. Let $\alpha_{j, k}, \beta_{j, k}$ be defined for positive integers $j, k$ such that $\alpha_{k, j}=$ $-\alpha_{j, k}, \beta_{k, j}=-\beta_{j, k}$ and $\beta_{j, k}=0$ for $|j-k| \neq 1$. Let

$$
\begin{equation*}
A_{n}=\left[\alpha_{j, k}\right]_{j, k=1, \cdots, n}, \quad B_{n}=\left[\beta_{j, k}\right]_{j, k=1, \cdots, n}, \quad B_{n}^{\prime}=\operatorname{diag}\left(B_{n-1}, 0\right) \tag{30}
\end{equation*}
$$

be $n \times n$ antisymmetric matrices. Let's denote

$$
D_{n}(\lambda):=\operatorname{det}\left(B_{n}+2 \lambda I_{n}\right), D_{0}(\lambda):=1
$$

where $I_{n}$ is the identity matrix, then we have

$$
\begin{equation*}
D_{n+1}(\lambda)=2 \lambda D_{n}(\lambda)+\beta_{n, n+1}^{2} D_{n-1}(\lambda) \text { for } n \in \mathbb{Z}, n>0 \tag{31}
\end{equation*}
$$

If $n>0$ is even, then we have (let's define $\operatorname{Pf}\left(B_{0}+\lambda A_{0}\right):=1$ )

$$
\begin{equation*}
\operatorname{Pf}\left(B_{n}+\lambda A_{n}\right)=\operatorname{Pf}\left(B_{n}^{\prime}+\lambda A_{n}\right)+\beta_{n-1, n} \operatorname{Pf}\left(B_{n-2}+\lambda A_{n-2}\right) \tag{32}
\end{equation*}
$$

Moreover, if $n>0$ is even and $B_{n} A_{n}=-4 I_{n}$, then we have

$$
\begin{equation*}
\operatorname{Pf}\left(B_{n}+\lambda^{2} A_{n}\right)=D_{n}(\lambda) /\left(\operatorname{Pf} B_{n}\right) \tag{33}
\end{equation*}
$$

and

$$
\begin{equation*}
\operatorname{Pf}\left(B_{n}^{\prime}+\lambda^{2} A_{n}\right)=2 \lambda D_{n-1}(\lambda) /\left(\operatorname{Pf} B_{n}\right) \tag{34}
\end{equation*}
$$

Proof. The formula (31) follows from the Laplace expansion of the determinant in the $(n+1)$-th row of $B_{n+1}+2 \lambda I_{n+1}$. The formula (32) follows from the Laplace expansion of the Pfaffian (see (6.36) in [18). Now we assume that $n>0$ is even and $B_{n} A_{n}=-4 I_{n}$, then $B_{n}$ is invertible, $A_{n}=-4 B_{n}^{-1}$ and

$$
\begin{aligned}
B_{n}+\lambda^{2} A_{n} & =B_{n}-4 \lambda^{2} B_{n}^{-1}=\left(B_{n}-2 \lambda I_{n}\right) B_{n}^{-1}\left(B_{n}+2 \lambda I_{n}\right) \\
& =-\left(B_{n}+2 \lambda I_{n}\right)^{T} B_{n}^{-1}\left(B_{n}+2 \lambda I_{n}\right)
\end{aligned}
$$

here we used the fact that $B_{n}$ is antisymmetric. By (28) we have

$$
\operatorname{Pf}\left(B_{n}+\lambda^{2} A_{n}\right)=(-1)^{n / 2} \operatorname{det}\left(B_{n}+2 \lambda I_{n}\right) \operatorname{Pf}\left(B_{n}^{-1}\right)
$$

Taking $\lambda=0$, we have $\operatorname{Pf}\left(B_{n}\right)=(-1)^{n / 2} \operatorname{det}\left(B_{n}\right) \operatorname{Pf}\left(B_{n}^{-1}\right)$. Since $B_{n}$ is invertible, by (28) again, we have $\operatorname{det}\left(B_{n}\right)=\left(\operatorname{Pf} B_{n}\right)^{2} \neq 0$, and thus $(-1)^{n / 2} \operatorname{Pf}\left(B_{n}^{-1}\right)=$ $\left(\operatorname{Pf} B_{n}\right)^{-1}$. Therefore, we have

$$
\operatorname{Pf}\left(B_{n}+\lambda^{2} A_{n}\right)=\operatorname{det}\left(B_{n}+2 \lambda I_{n}\right)\left(\operatorname{Pf} B_{n}\right)^{-1}=D_{n}(\lambda) /\left(\operatorname{Pf} B_{n}\right)
$$

which is (33). By definition, the above result is also true for $n=0$. By definition of a Pfaffian and the fact that $\beta_{j, k}=0$ for $|j-k| \neq 1$, we have

$$
\operatorname{Pf} B_{n}=\prod_{j=1}^{n / 2} \beta_{2 j-1,2 j}, \operatorname{Pf} B_{n}=\beta_{n-1, n} \operatorname{Pf} B_{n-2}
$$

Combining this with (31), (32) and (33), we have

$$
\begin{aligned}
& \operatorname{Pf}\left(B_{n}^{\prime}+\lambda^{2} A_{n}\right)=\operatorname{Pf}\left(B_{n}+\lambda^{2} A_{n}\right)-\beta_{n-1, n} \operatorname{Pf}\left(B_{n-2}+\lambda^{2} A_{n-2}\right) \\
& =D_{n}(\lambda) /\left(\operatorname{Pf} B_{n}\right)-\beta_{n-1, n} D_{n-2}(\lambda) /\left(\operatorname{Pf} B_{n-2}\right) \\
& =D_{n}(\lambda) /\left(\operatorname{Pf} B_{n}\right)-\beta_{n-1, n}^{2} D_{n-2}(\lambda) /\left(\operatorname{Pf} B_{n}\right)=2 \lambda D_{n-1}(\lambda) /\left(\operatorname{Pf} B_{n}\right),
\end{aligned}
$$

which is (34). This completes the proof.
We also need to evaluate the determinant $D_{n}(\lambda)$.
Lemma 6. Let $\beta_{j, k}$ be defined in Lemma 圆, i.e. $\beta_{j, k}$ satisfies (29). Let's denote $B_{n}=\left[\beta_{j, k}\right]_{j, k=1, \cdots, n}$ and $D_{n}(\lambda)=\operatorname{det}\left(B_{n}+2 \lambda I_{n}\right)$ with $D_{0}(\lambda)=1$, then we have

$$
D_{n}(\lambda)=\sum_{m=0}^{[n / 2]} 2^{n-m}\binom{n}{2 m} \frac{(2 m)!}{2^{m} m!} \lambda^{n-2 m}
$$

Proof. By (29) and (31), we have

$$
D_{n+1}(\lambda)=2 \lambda D_{n}(\lambda)+2 n D_{n-1}(\lambda) \text { for } n \in \mathbb{Z}, n>0
$$

Let $\widetilde{H}_{n}(x)=i^{-n} D_{n}(i x)$, then we have

$$
\widetilde{H}_{n+1}(x)=2 x \widetilde{H}_{n}(x)-2 n \widetilde{H}_{n-1}(x) \text { for } n \in \mathbb{Z}, n>0
$$

Moreover, we have $D_{0}(\lambda)=1, B_{1}=(0), D_{1}(\lambda)=2 \lambda ; \quad \widetilde{H}_{0}(x)=1, \widetilde{H}_{1}(x)=2 x$. Thus $\widetilde{H}_{n}$ satisfy the same iteration formula and initial condition as the Hermite polynomials $H_{n}$ (recall (19)), which implies that $\widetilde{H}_{n}(x)=H_{n}(x)$. By (21) we have

$$
\begin{aligned}
& D_{n}(\lambda)=i^{n} \widetilde{H}_{n}(-i \lambda)=i^{n} H_{n}(-i \lambda) \\
= & \sum_{m=0}^{[n / 2]} i^{n}(-1)^{m} 2^{n-m}\binom{n}{2 m} \frac{(2 m)!}{2^{m} m!}(-i \lambda)^{n-2 m} \\
= & \sum_{m=0}^{[n / 2]} 2^{n-m}\binom{n}{2 m} \frac{(2 m)!}{2^{m} m!} \lambda^{n-2 m},
\end{aligned}
$$

which completes the proof.
Now we give the proof of Lemma 1
Proof. Let $\alpha_{j, k}, \beta_{j, k}$ be defined in Lemma 2, $\nu_{k}$ be defined in Lemma 3, and $A_{n}, B_{n}, B_{n}^{\prime}$ be defined in (30). If $n$ is even, then by (e) of Lemma 4 we have $B_{n} A_{n}=-4 I_{n}$. By Lemma 2, Lemma 5 and Lemma 6e have

$$
G_{n-2 k, k}=(n-2 k)!k!c_{n}\left[\zeta^{n / 2-k}\right] \operatorname{Pf}\left[\beta_{j, l}+\zeta \alpha_{j, l}\right]_{j, l=1, \cdots, n}
$$

$$
\begin{aligned}
& =(n-2 k)!k!c_{n}\left[\zeta^{n / 2-k}\right] \operatorname{Pf}\left(B_{n}+\zeta A_{n}\right) \\
& =(n-2 k)!k!c_{n}\left[\zeta^{n-2 k}\right] \operatorname{Pf}\left(B_{n}+\zeta^{2} A_{n}\right) \\
& =(n-2 k)!k!c_{n}\left[\zeta^{n-2 k}\right] D_{n}(\zeta) /\left(\operatorname{Pf} B_{n}\right) \\
& =(n-2 k)!k!c_{n} 2^{n-k}\binom{n}{2 k} \frac{(2 k)!}{2^{k} k!}\left(\operatorname{Pf} B_{n}\right)^{-1} \\
& =c_{n} 2^{n-2 k} n!\left(\operatorname{Pf} B_{n}\right)^{-1}
\end{aligned}
$$

and thus

$$
G_{n-2 k, k}=2^{-2 k} G_{n, 0}=2^{-2 k} G_{n} .
$$

If $n$ is odd, by Lemma 3, we first have

$$
\frac{G_{n-2 k, k}}{(n-2 k)!k!c_{n}}=\left[\zeta^{(n-2 k-1) / 2}\right] \operatorname{Pf}\left[\begin{array}{ll}
{\left[\beta_{j, l}+\zeta \alpha_{j, l}\right]_{j, l=1, \cdots, n}} & {\left[\nu_{j}\right]_{j=1, \cdots, n}} \\
-\left[\nu_{l}\right]_{l=1, \cdots, n} & 0
\end{array}\right]
$$

By Lemma 4. we also have $B_{n+1} A_{n+1}=-4 I_{n+1}, \alpha_{j, n+1}=\alpha_{1, n+1} \nu_{j} / \nu_{1}=-\alpha_{n+1, j}$ for $0<j \leq n$, and $\alpha_{1, n+1}>0, \nu_{1}>0$. By definition, $\operatorname{Pf} X$ is linear with respect to the last row of $X$, thus for $\lambda:=\zeta \alpha_{1, n+1} / \nu_{1}$, we have

$$
\begin{aligned}
& \lambda \operatorname{Pf}\left[\begin{array}{ll}
{\left[\beta_{j, l}+\zeta \alpha_{j, l}\right]_{j, l=1, \cdots, n}} & {\left[\nu_{j}\right]_{j=1, \cdots, n}} \\
-\left[\nu_{l}\right]_{l=1, \cdots, n} & 0
\end{array}\right] \\
= & \operatorname{Pf}\left[\begin{array}{ll}
{\left[\beta_{j, l}+\zeta \alpha_{j, l}\right]_{j, l=1, \cdots, n}} & \lambda\left[\nu_{j}\right]_{j=1, \cdots, n} \\
-\lambda\left[\nu_{l}\right]_{l=1, \cdots, n} & 0
\end{array}\right] \\
= & \operatorname{Pf}\left[\begin{array}{ll}
{\left[\beta_{j, l}+\zeta \alpha_{j, l}\right]_{j, l=1, \cdots, n}} & {\left[\zeta \alpha_{j, n+1}\right]_{j=1, \cdots, n}} \\
{\left[\zeta \alpha_{n+1, l}\right]_{l=1, \cdots, n}} & 0
\end{array}\right] \\
= & \operatorname{Pf}\left(B_{n+1}^{\prime}+\zeta A_{n+1}\right),
\end{aligned}
$$

where $B_{n+1}^{\prime}=\operatorname{diag}\left(B_{n}, 0\right)$. Hence, by Lemma 5 and Lemma 6, we have

$$
\begin{aligned}
& \frac{\alpha_{1, n+1} G_{n-2 k, k}}{\nu_{1}(n-2 k)!k!c_{n}} \\
= & \frac{\alpha_{1, n+1}}{\nu_{1}}\left[\zeta^{(n-2 k-1) / 2+1}\right] \zeta \operatorname{Pf}\left[\begin{array}{cc}
{\left[\beta_{j, l}+\zeta \alpha_{j, l}\right]_{j, l=1, \cdots, n}} & {\left[\nu_{j}\right]_{j=1, \cdots, n}} \\
-\left[\nu_{l}\right]_{l=1, \cdots, n} & 0
\end{array}\right] \\
= & {\left[\zeta^{(n-2 k-1) / 2+1}\right] \lambda \operatorname{Pf}\left[\begin{array}{cc}
{\left[\beta_{j, l}+\zeta \alpha_{j, l}\right]_{j, l=1, \cdots, n}} & {\left[\nu_{j}\right]_{j=1, \cdots, n}} \\
-\left[\nu_{l}\right]_{l=1, \cdots, n} & 0
\end{array}\right] } \\
= & {\left[\zeta^{(n-2 k-1) / 2+1}\right] \operatorname{Pf}\left(B_{n+1}^{\prime}+\zeta A_{n+1}\right) } \\
= & {\left[\zeta^{n-2 k+1}\right] \operatorname{Pf}\left(B_{n+1}^{\prime}+\zeta^{2} A_{n+1}\right) } \\
= & {\left[\zeta^{n-2 k+1}\right]\left(2 \zeta D_{n}(\zeta) /\left(\operatorname{Pf} B_{n+1}\right)\right) } \\
= & 2\left[\zeta^{n-2 k}\right] D_{n}(\zeta) /\left(\operatorname{Pf} B_{n+1}\right) \\
= & \frac{2^{n-k+1}}{\operatorname{Pf} B_{n+1}}\binom{n}{2 k} \frac{(2 k)!}{2^{k} k!} .
\end{aligned}
$$

Therefore, we have

$$
G_{n-2 k, k}=\frac{2^{n-2 k+1} n!\nu_{1} c_{n}}{\alpha_{1, n+1} \operatorname{Pf} B_{n+1}}
$$

which implies

$$
G_{n-2 k, k}=2^{-2 k} G_{n, 0}=2^{-2 k} G_{n} .
$$

This completes the whole proof of Lemma 1

## 4. Auxiliary point processes

We need to introduce two more auxiliary point processes to derive the main result. First, instead of $\chi^{(n)}$ (recall (8)), it is more convenient to consider the point process defined as

$$
\widetilde{\chi}^{(n)}=\sum_{i<j} \delta_{n\left|\lambda_{i}-\lambda_{j}\right|}=\sum_{\lambda_{i}>\lambda_{j}} \delta_{n\left(\lambda_{i}-\lambda_{j}\right)} .
$$

Then we have

$$
\chi^{(n)} \leq \widetilde{\chi}^{(n)}
$$

in fact, we can write

$$
\widetilde{\chi}^{(n)}=\sum_{j=1}^{n-1} \widetilde{\chi}^{(n, j)}
$$

such that

$$
\widetilde{\chi}^{(n, j)}=\sum_{i=1}^{n-j} \delta_{n\left(\lambda_{(i+j)}-\lambda_{(i)}\right)}
$$

For any Borel set $B \subset \mathbb{R}$, we have

$$
\widetilde{\chi}^{(n, 1)}=\chi^{(n)} \text { and } 0 \leq \widetilde{\chi}^{(n, j)}(B) \leq n
$$

For the auxiliary point process $\tilde{\chi}^{(n)} \geq \chi^{(n)}$, we will prove that $\tilde{\chi}^{(n)}-\chi^{(n)} \rightarrow 0$ as $n \rightarrow \infty$ almost surely (see Lemma 8), which indicates that there is no successive small gaps.

We now introduce another auxiliary point process as

$$
\rho^{(k, n)}=\sum_{i_{1}, \cdots, i_{2 k}} \sum_{\text {all distinct, } i_{2 j-1}<i_{2 j}} \delta_{\left(n\left|\lambda_{i_{1}}-\lambda_{i_{2}}\right|, \cdots, n\left|\lambda_{i_{2 k-1}}-\lambda_{i_{2 k}}\right|\right)} .
$$

The following lemma gives the estimates of $\rho^{(k, n)}$ in terms of $\widetilde{\chi}^{(n)}$, and we will see that $\rho^{(k, n)}$ is basically equivalent to the factorial moment of $\widetilde{\chi}^{(n)}$ (see (63)).
Lemma 7. For any bounded interval $A \subset \mathbb{R}_{+}$, we have

$$
\begin{equation*}
\rho^{(k, n)}\left(A^{k}\right) \leq \frac{\left(\tilde{\chi}^{(n)}(A)\right)!}{\left(\widetilde{\chi}^{(n)}(A)-k\right)!} \tag{35}
\end{equation*}
$$

Given $c_{1}$ such that $A \subset\left(0, c_{1}\right)$, let's denote $c_{n}=c_{1} n^{-1}$ and

$$
\begin{equation*}
a=\max \left\{i-j: i, j \in \mathbb{Z} \cap[1, n], \lambda_{(i)}-\lambda_{(j)}<2 c_{n}\right\} \tag{36}
\end{equation*}
$$

If $c_{n} \in(0,1)$, then we have

$$
\begin{equation*}
0 \leq \frac{\left(\widetilde{\chi}^{(n)}(A)\right)!}{\left(\widetilde{\chi}^{(n)}(A)-k\right)!}-\rho^{(k, n)}\left(A^{k}\right) \leq k(k-1)(a-1)\left(\widetilde{\chi}^{(n)}(A)\right)^{k-1} \tag{37}
\end{equation*}
$$

and

$$
\begin{equation*}
\rho^{(k, n)}\left(A^{k}\right) \geq\left(\widetilde{\chi}^{(n)}(A)\right)^{k}-k(k-1) a\left(\widetilde{\chi}^{(n)}(A)\right)^{k-1} \tag{38}
\end{equation*}
$$

Moreover, let $A_{1}=\left(0,2 c_{1}\right)$, then we have

$$
\begin{equation*}
\rho^{(k, n)}\left(A_{1}^{k}\right) \geq \frac{(a+1)!}{(a+1-2 k)!2^{k}} \tag{39}
\end{equation*}
$$

Proof. Let's denote

$$
\begin{aligned}
& X_{1}=\left\{\left(i_{1}, \cdots, i_{2 k}\right): i_{j} \in \mathbb{Z}, 1 \leq i_{j} \leq n, \forall 1 \leq j \leq 2 k\right. \\
& \left.i_{2 j-1}<i_{2 j}, \forall 1 \leq j \leq k, \quad\left\{i_{2 j-1}, i_{2 j}\right\} \neq\left\{i_{2 l-1}, i_{2 l}\right\}, \forall 1 \leq j<l \leq k\right\} \\
& X_{2}=\left\{\left(i_{1}, \cdots, i_{2 k}\right): i_{j} \in \mathbb{Z}, 1 \leq i_{j} \leq n, \forall 1 \leq j \leq 2 k\right. \\
& \left.i_{2 j-1}<i_{2 j}, \forall 1 \leq j \leq k, \quad i_{j} \neq i_{l}, \forall 1 \leq j<l \leq 2 k\right\} \\
& Y_{j, l}=\left\{\left(i_{1}, \cdots, i_{2 k}\right) \in X_{1}:\left\{i_{2 j-1}, i_{2 j}\right\} \cap\left\{i_{2 l-1}, i_{2 l}\right\} \neq \emptyset\right\}
\end{aligned}
$$

then we have

$$
X_{2} \subseteq X_{1} \text { and } X_{1} \backslash X_{2}=\cup_{1 \leq j<l \leq k} Y_{j, l}
$$

Let

$$
\begin{aligned}
& X_{m, A}=\left\{\left(i_{1}, \cdots, i_{2 k}\right) \in X_{m}: n\left|\lambda_{i_{2 j-1}}-\lambda_{i_{2 j}}\right| \in A, \forall 1 \leq j \leq k\right\}, m=1,2, \\
& Y_{m, l, A}=\left\{\left(i_{1}, \cdots, i_{2 k}\right) \in Y_{m, l}: n\left|\lambda_{i_{2 j-1}}-\lambda_{i_{2 j}}\right| \in A, \forall 1 \leq j \leq k\right\},
\end{aligned}
$$

then we have

$$
\rho^{(k, n)}\left(A^{k}\right)=\left|X_{2, A}\right|, \quad X_{2, A} \subseteq X_{1, A} \text { and }\left|X_{1, A}\right|=\frac{\left(\widetilde{\chi}^{(n)}(A)\right)!}{\left(\widetilde{\chi}^{(n)}(A)-k\right)!}
$$

which implies (35), here $|X|$ is cardinality of the set $X$.
We also have $X_{1, A} \backslash X_{2, A}=\cup_{1 \leq j<l \leq k} Y_{j, l, A}$ and $\left|Y_{j, l, A}\right|=\left|Y_{1,2, A}\right|$ for $1 \leq j<$ $l \leq k$ by symmetry. Therefore, we have

$$
\begin{equation*}
\left|X_{1, A}\right|-\left|X_{2, A}\right| \leq \sum_{1 \leq j<l \leq k}\left|Y_{j, l, A}\right|=k(k-1)\left|Y_{1,2, A}\right| / 2 . \tag{40}
\end{equation*}
$$

If $a=0$, then we have $n\left|\lambda_{j}-\lambda_{l}\right| \geq n\left(2 c_{n}\right)=2 c_{1}$ for every $1 \leq j<l \leq n$, i.e., $n\left|\lambda_{j}-\lambda_{l}\right| \notin A$, and thus $\widetilde{\chi}^{(n)}(A)=\rho^{(k, n)}\left(A^{k}\right)=0$; if $k=1$ then $\widetilde{\chi}^{(n)}(A)=\rho^{(1, n)}(A)$ by definitions. In both cases, the inequalities (37) and (38) are clearly true, thus we only need to consider the case $a>0, k>1$.

Let $\lambda_{i, j}:=n\left|\lambda_{i}-\lambda_{j}\right|$. For fixed $\lambda_{i_{1}, i_{2}} \in A$, let

$$
\begin{aligned}
T_{j} & =\left\{l: l \neq i_{j}, n\left|\lambda_{i_{l}}-\lambda_{i_{j}}\right| \in A\right\} \\
T_{j}^{\prime} & =\left\{l: l \neq i_{j},\left|\lambda_{i_{l}}-\lambda_{i_{j}}\right| \in\left(0, c_{n}\right)\right\}, j=1,2 .
\end{aligned}
$$

Then we have $T_{j} \subseteq T_{j}^{\prime}$ because $n\left|\lambda_{i_{l}}-\lambda_{i_{j}}\right| \in A$ implies $\left|\lambda_{i_{l}}-\lambda_{i_{j}}\right| \in n^{-1} A \subset$ $n^{-1}\left(0, c_{1}\right)=\left(0, c_{n}\right)$. Let's assume $\lambda_{i_{1}}=\lambda_{(p)}$, then we have

$$
\begin{aligned}
\left\{\lambda_{l}: l \in T_{1}^{\prime} \cup\left\{i_{1}\right\}\right\} & =\left\{\lambda_{(q)}:\left|\lambda_{(q)}-\lambda_{(p)}\right|<c_{n}\right\} \\
& =\left\{\lambda_{(q)}: r \leq q \leq s\right\},
\end{aligned}
$$

for some $r, s \in \mathbb{Z} \cap[1, n]$ such that $\left|\lambda_{(r)}-\lambda_{(p)}\right|<c_{n},\left|\lambda_{(s)}-\lambda_{(p)}\right|<c_{n}$. Therefore, we have $\left|\lambda_{(r)}-\lambda_{(s)}\right|<2 c_{n}$ and $s-r \leq a$ by the definition of $a$. Since $i_{1} \notin T_{1}^{\prime}$, we have

$$
\begin{aligned}
\left|T_{1}^{\prime}\right|+1 & =\left|\left\{\lambda_{l}: l \in T_{1}^{\prime} \cup\left\{i_{1}\right\}\right\}\right|=\left|\left\{\lambda_{(q)}: r \leq q \leq s\right\}\right| \\
& \leq s-r+1 \leq a+1
\end{aligned}
$$

and thus $\left|T_{1}\right| \leq\left|T_{1}^{\prime}\right| \leq a$. Similarly we have $\left|T_{2}\right| \leq\left|T_{2}^{\prime}\right| \leq a$.
Now for $\lambda_{i_{1}, i_{2}} \in A$, by definition we have $i_{2} \in T_{1}$ and $i_{1} \in T_{2}$. If $\lambda_{i_{3}, i_{4}} \in A, i_{3}<$ $i_{4},\left\{i_{1}, i_{2}\right\} \cap\left\{i_{3}, i_{4}\right\} \neq \emptyset,\left\{i_{1}, i_{2}\right\} \neq\left\{i_{3}, i_{4}\right\}$, then we must have $\left\{i_{3}, i_{4}\right\}=\left\{i_{1}, l\right\}, l \in$ $T_{2} \backslash\left\{i_{1}\right\}$ or $\left\{i_{3}, i_{4}\right\}=\left\{i_{2}, l\right\}, l \in T_{1} \backslash\left\{i_{2}\right\}$. Thus for $\lambda_{i_{1}, i_{2}} \in A$, the number of $\left(i_{3}, i_{4}\right)$ satisfying $\lambda_{i_{3}, i_{4}} \in A, i_{3}<i_{4},\left\{i_{1}, i_{2}\right\} \cap\left\{i_{3}, i_{4}\right\} \neq \emptyset,\left\{i_{1}, i_{2}\right\} \neq\left\{i_{3}, i_{4}\right\}$ is at
most $\left|T_{2} \backslash\left\{i_{1}\right\}\right|+\left|T_{1} \backslash\left\{i_{2}\right\}\right|=\left|T_{2}\right|-1+\left|T_{1}\right|-1 \leq 2(a-1)$. Now there are $\tilde{\chi}^{(n)}(A)$ choices of $\left(i_{1}, i_{2}\right)$; for fixed $\left(i_{1}, i_{2}\right)$, there are at most $2(a-1)$ choices of $\left(i_{3}, i_{4}\right)$ and $\widetilde{\chi}^{(n)}(A)$ choices of $\left(i_{2 l-1}, i_{2 l}\right)$ with $3 \leq l \leq k$ to satisfy $\left(i_{1}, \cdots, i_{2 k}\right) \in Y_{1,2, A}$, thus we have

$$
\left|Y_{1,2, A}\right| \leq \tilde{\chi}^{(n)}(A) \times 2(a-1) \times \tilde{\chi}^{(n)}(A)^{k-2}=2(a-1) \tilde{\chi}^{(n)}(A)^{k-1}
$$

Therefore, by (40) we have

$$
\begin{aligned}
0 & \leq \frac{\left(\widetilde{\chi}^{(n)}(A)\right)!}{\left(\widetilde{\chi}^{(n)}(A)-k\right)!}-\rho^{(k, n)}\left(A^{k}\right) \\
& =\left|X_{1, A}\right|-\left|X_{2, A}\right| \\
& \leq k(k-1)\left|Y_{1,2, A}\right| / 2 \\
& \leq k(k-1)(a-1)\left(\widetilde{\chi}^{(n)}(A)\right)^{k-1}
\end{aligned}
$$

which is (37). The inequality (38) follows from (37) and the fact that

$$
\begin{aligned}
\frac{\left(\widetilde{\chi}^{(n)}(A)\right)!}{\left(\widetilde{\chi}^{(n)}(A)-k\right)!} & =\prod_{j=0}^{k-1}\left(\widetilde{\chi}^{(n)}(A)-j\right)=\left(\widetilde{\chi}^{(n)}(A)\right)^{k} \prod_{j=0}^{k-1}\left(1-j / \widetilde{\chi}^{(n)}(A)\right) \\
& \geq\left(\widetilde{\chi}^{(n)}(A)\right)^{k}\left(1-\sum_{j=0}^{k-1} j / \widetilde{\chi}^{(n)}(A)\right) \\
& =\left(\widetilde{\chi}^{(n)}(A)\right)^{k}-k(k-1)\left(\widetilde{\chi}^{(n)}(A)\right)^{k-1} / 2 .
\end{aligned}
$$

To prove (39), by the definition of $a$, there exists $r, s \in \mathbb{Z} \cap[1, n]$ such that $\mid \lambda_{(r)}-$ $\lambda_{(s)} \mid<2 c_{n}, s-r=a$. Let

$$
Z=\left\{j: \lambda_{j}=\lambda_{(q)}, r \leq q \leq s\right\}
$$

and

$$
\begin{aligned}
& X_{3}=\left\{\left(i_{1}, \cdots, i_{2 k}\right): i_{j} \in Z, \forall 1 \leq j \leq 2 k\right. \\
& \left.i_{2 j-1}<i_{2 j}, \forall 1 \leq j \leq k, \quad i_{j} \neq i_{l}, \forall 1 \leq j<l \leq 2 k\right\}
\end{aligned}
$$

Then we have $|Z|=s-r+1=a+1, X_{3} \subseteq X_{2}$ and

$$
\left|X_{3}\right|=\frac{|Z|!}{(|Z|-2 k)!2^{k}}=\frac{(a+1)!}{(a+1-2 k)!2^{k}}
$$

Moreover, we have $\left|\lambda_{j}-\lambda_{l}\right| \leq\left|\lambda_{(r)}-\lambda_{(s)}\right|<2 c_{n}$ for $j, l \in Z$. For $\left(i_{1}, \cdots, i_{2 k}\right) \in X_{3}$, we have $0<n\left|\lambda_{i_{2 j-1}}-\lambda_{i_{2 j}}\right|<2 n c_{n}=2 c_{1}$ for $1 \leq j \leq k$, i.e., $n\left|\lambda_{i_{2 j-1}}-\lambda_{i_{2 j}}\right| \in$ $\left(0,2 c_{1}\right)=A_{1}$. Therefore, we have $X_{3} \subseteq X_{2, A_{1}}$ and thus

$$
\rho^{(k, n)}\left(A_{1}^{k}\right)=\left|X_{2, A_{1}}\right| \geq\left|X_{3}\right|=\frac{(a+1)!}{(a+1-2 k)!2^{k}}
$$

which is (39). This completes the whole proof.

## 5. No SUCCESSIVE SMALL GAPS

In this section, we will prove the following lemma which indicates that there is no successive small gaps.
Lemma 8. For any bounded interval $A \subset \mathbb{R}_{+}$, we have $\chi^{(n)}(A)-\widetilde{\chi}^{(n)}(A) \rightarrow 0$ in probability as $n \rightarrow+\infty$.

To prove Lemma 8 , we will need the upper and lower bounds in the following integral lemma.

Lemma 9. Let's assume $\lambda_{j} \in \mathbb{R}$ (not necessarily distinct) for $1 \leq j \leq m$, $m$ and $n$ are positive integers such that $m<n$, and $2 n c^{2} \in(0,1)$ with $c>0$. Let's denote

$$
\begin{equation*}
F(x):=e^{-x^{2} / 2} \prod_{j=1}^{m}\left(x-\lambda_{j}\right) \tag{41}
\end{equation*}
$$

then we have

$$
\begin{equation*}
\int_{\mathbb{R}}\left|F^{\prime}(x)\right|^{2} d x \leq(2 n) \int_{\mathbb{R}}|F(x)|^{2} d x \tag{42}
\end{equation*}
$$

and

$$
\begin{align*}
\left(1-n c^{2}\right) c^{2} \int_{\mathbb{R}} d x_{1}\left|F\left(x_{1}\right)\right|^{2} & \leq \int_{\mathbb{R}} d x_{1} \int_{x_{1}-c}^{x_{1}+c} d x_{2}\left|x_{1}-x_{2}\right|\left|F\left(x_{1}\right)\right|\left|F\left(x_{2}\right)\right|  \tag{43}\\
& \leq c^{2} \int_{\mathbb{R}} d x_{1}\left|F\left(x_{1}\right)\right|^{2}
\end{align*}
$$

Moreover, given an interval $A \subset(0, c)$, let's denote $A_{1}=A \cup(-A)$ and

$$
\begin{equation*}
\varphi(A):=\int_{A} u d u \tag{44}
\end{equation*}
$$

then we have

$$
\begin{align*}
\left(1-n c^{2}\right) \cdot 2 \varphi(A) \int_{\mathbb{R}} d x_{1}\left|F\left(x_{1}\right)\right|^{2} & \leq \int_{\mathbb{R}} d x_{1} \int_{x_{1}+A_{1}} d x_{2}\left|x_{1}-x_{2}\right|\left|F\left(x_{1}\right)\right|\left|F\left(x_{2}\right)\right| \\
& \leq 2 \varphi(A) \int_{\mathbb{R}} d x_{1}\left|F\left(x_{1}\right)\right|^{2} \tag{45}
\end{align*}
$$

Given $B=\cup_{i=1}^{m}\left(\lambda_{i}, \lambda_{i}+c\right)^{2} \subset \mathbb{R}^{2}$, we have

$$
\begin{equation*}
\int_{B}\left|x_{1}-x_{2}\right|\left|F\left(x_{1}\right)\right|\left|F\left(x_{2}\right)\right| d x_{1} d x_{2} \leq n c^{4} \int_{\mathbb{R}}|F(x)|^{2} d x \tag{46}
\end{equation*}
$$

Proof. Note that $F(x) \in V_{m}$ (see (23)), therefore, we can write

$$
F(x)=\sum_{j=0}^{m} a_{j} \varphi_{j}(x)
$$

By (20) we have

$$
\begin{aligned}
F^{\prime}(x) & =\sum_{j=0}^{m} \frac{a_{j}}{\sqrt{2}}\left(\sqrt{j} \varphi_{j-1}(x)-\sqrt{j+1} \varphi_{j+1}(x)\right) \\
& =\sum_{j=0}^{m+1} \frac{\sqrt{j+1} a_{j+1}-\sqrt{j} a_{j-1}}{\sqrt{2}} \varphi_{j}(x)
\end{aligned}
$$

where $\varphi_{-1}(x)=0, a_{-1}=a_{m+1}=a_{m+2}=0$. By (18) we have

$$
\int_{\mathbb{R}}|F(x)|^{2} d x=\sum_{j=0}^{m}\left|a_{j}\right|^{2}
$$

and

$$
\int_{\mathbb{R}}\left|F^{\prime}(x)\right|^{2} d x=\sum_{j=0}^{m+1}\left|\frac{\sqrt{j+1} a_{j+1}-\sqrt{j} a_{j-1}}{\sqrt{2}}\right|^{2}
$$

Using $|a+b|^{2} \leq 2\left(|a|^{2}+|b|^{2}\right)$ and $a_{-1}=a_{m+1}=a_{m+2}=0$, we have

$$
\begin{aligned}
\int_{\mathbb{R}}\left|F^{\prime}(x)\right|^{2} d x & \leq \sum_{j=0}^{m+1}\left(\left|\sqrt{j+1} a_{j+1}\right|^{2}+\left|\sqrt{j} a_{j-1}\right|^{2}\right) \\
& =\sum_{j=1}^{m+2} j\left|a_{j}\right|^{2}+\sum_{j=-1}^{m}(j+1)\left|a_{j}\right|^{2}=\sum_{j=0}^{m}(2 j+1)\left|a_{j}\right|^{2} \\
& \leq \sum_{j=0}^{m}(2 m+1)\left|a_{j}\right|^{2}=(2 m+1) \int_{\mathbb{R}}|F(x)|^{2} d x \\
& \leq(2 n) \int_{\mathbb{R}}|F(x)|^{2} d x
\end{aligned}
$$

which is the first inequality (42). Here we used the fact that $m<n, n \geq m+1$.
To prove (43), a change of variables $x_{2}=x_{1}+t$ yields

$$
\begin{align*}
& \int_{\mathbb{R}} d x_{1} \int_{x_{1}-c}^{x_{1}+c} d x_{2}\left|x_{1}-x_{2}\right|\left|F\left(x_{1}\right)\right|\left|F\left(x_{2}\right)\right|  \tag{47}\\
= & \int_{-c}^{c}|t| d t \int_{\mathbb{R}}\left|F\left(x_{1}\right)\right|\left|F\left(x_{1}+t\right)\right| d x_{1} .
\end{align*}
$$

We also have

$$
\begin{align*}
& \int_{\mathbb{R}} \| F\left(x_{1}\right)\left|-\left|F\left(x_{1}+t\right)\right|^{2} d x_{1}\right. \\
= & \int_{\mathbb{R}}\left(\left|F\left(x_{1}\right)\right|^{2}+\left|F\left(x_{1}+t\right)\right|^{2}\right) d x_{1}-2 \int_{\mathbb{R}}\left|F\left(x_{1}\right)\right|\left|F\left(x_{1}+t\right)\right| d x_{1}  \tag{48}\\
= & 2 \int_{\mathbb{R}}\left|F\left(x_{1}\right)\right|^{2} d x_{1}-2 \int_{\mathbb{R}}\left|F\left(x_{1}\right)\right|\left|F\left(x_{1}+t\right)\right| d x_{1},
\end{align*}
$$

which implies

$$
\begin{equation*}
\int_{\mathbb{R}}\left|F\left(x_{1}\right)\right|\left|F\left(x_{1}+t\right)\right| d x_{1} \leq \int_{\mathbb{R}}\left|F\left(x_{1}\right)\right|^{2} d x_{1} \tag{49}
\end{equation*}
$$

By (47) and (49), we have

$$
\begin{aligned}
& \int_{\mathbb{R}} d x_{1} \int_{x_{1}-c}^{x_{1}+c} d x_{2}\left|x_{1}-x_{2}\right|\left|F\left(x_{1}\right)\right|\left|F\left(x_{2}\right)\right| \\
\leq & \int_{-c}^{c}|t| d t \int_{\mathbb{R}}\left|F\left(x_{1}\right)\right|^{2} d x_{1}=c^{2} \int_{\mathbb{R}}\left|F\left(x_{1}\right)\right|^{2} d x_{1}
\end{aligned}
$$

which is the upper bound in (43).
On the other hand, we have

$$
\begin{aligned}
& \left|\left|F\left(x_{1}\right)\right|-\left|F\left(x_{1}+t\right)\right|^{2} \leq\left|F\left(x_{1}\right)-F\left(x_{1}+t\right)\right|^{2}\right. \\
= & \left|-t \int_{0}^{1} F^{\prime}\left(x_{1}+t s\right) d s\right|^{2} \leq|t|^{2} \int_{0}^{1}\left|F^{\prime}\left(x_{1}+t s\right)\right|^{2} d s
\end{aligned}
$$

and thus by (42) we have

$$
\begin{aligned}
& \int_{\mathbb{R}}\left\|F\left(x_{1}\right)|-| F\left(x_{1}+t\right)\right\|^{2} d x_{1} \\
\leq & |t|^{2} \int_{\mathbb{R}} \int_{0}^{1}\left|F^{\prime}\left(x_{1}+t s\right)\right|^{2} d s d x_{1} \\
= & |t|^{2} \int_{0}^{1}\left[\int_{\mathbb{R}}\left|F^{\prime}\left(x_{1}+t s\right)\right|^{2} d x_{1}\right] d s=|t|^{2} \int_{0}^{1} \int_{\mathbb{R}}\left|F^{\prime}\left(x_{1}\right)\right|^{2} d x_{1} d s \\
= & |t|^{2} \int_{\mathbb{R}}\left|F^{\prime}\left(x_{1}\right)\right|^{2} d x_{1} \leq 2 n|t|^{2} \int_{\mathbb{R}}\left|F\left(x_{1}\right)\right|^{2} d x_{1} .
\end{aligned}
$$

Combining this estimate with identity (48), we have

$$
\int_{\mathbb{R}}\left|F\left(x_{1}\right)\right|\left|F\left(x_{1}+t\right)\right| d x_{1} \geq\left(1-n|t|^{2}\right) \int_{\mathbb{R}}\left|F\left(x_{1}\right)\right|^{2} d x_{1}, \quad \forall t \in(-c, c)
$$

and thus the uniform lower bound

$$
\begin{equation*}
\int_{\mathbb{R}}\left|F\left(x_{1}\right)\right|\left|F\left(x_{1}+t\right)\right| d x_{1} \geq\left(1-n c^{2}\right) \int_{\mathbb{R}}\left|F\left(x_{1}\right)\right|^{2} d x_{1}, \forall t \in(-c, c) \tag{50}
\end{equation*}
$$

Therefore, combining (47) and (50), we can conclude the lower bound in (43).
Notice that

$$
\begin{aligned}
& \int_{\mathbb{R}} d x_{1} \int_{x_{1}+A_{1}} d x_{2}\left|x_{1}-x_{2}\right|\left|F\left(x_{1}\right)\right|\left|F\left(x_{2}\right)\right| \\
= & \int_{A_{1}}|t| d t \int_{\mathbb{R}}\left|F\left(x_{1}\right)\right|\left|F\left(x_{1}+t\right)\right| d x_{1},
\end{aligned}
$$

then (45) follows from (49), (50) and the fact that

$$
\int_{A_{1}}|t| d t=2 \int_{A} t d t=2 \varphi(A)
$$

Let $B_{1}=\cup_{i=1}^{m}\left(\lambda_{i}, \lambda_{i}+c\right) \subset \mathbb{R}$, then for $\left(x_{1}, x_{2}\right) \in B=\cup_{i=1}^{m}\left(\lambda_{i}, \lambda_{i}+c\right)^{2}$, we have $x_{1}, x_{2} \in B_{1}, \quad\left(x_{2}, x_{1}\right) \in B,\left|x_{1}-x_{2}\right| \leq c$, and thus we first have

$$
\begin{aligned}
& \int_{B}\left|x_{1}-x_{2}\right|\left|F\left(x_{1}\right)\right|\left|F\left(x_{2}\right)\right| d x_{1} d x_{2} \leq \frac{1}{2} \int_{B}\left|x_{1}-x_{2}\right|\left(\left|F\left(x_{1}\right)\right|^{2}+\left|F\left(x_{2}\right)\right|^{2}\right) d x_{1} d x_{2} \\
= & \int_{B}\left|x_{1}-x_{2}\right|\left|F\left(x_{1}\right)\right|^{2} d x_{1} d x_{2} \leq \int_{B_{1}} d x_{1} \int_{x_{1}-c}^{x_{1}+c} d x_{2}\left|x_{1}-x_{2}\right|\left|F\left(x_{1}\right)\right|^{2} \\
= & c^{2} \int_{B_{1}}\left|F\left(x_{1}\right)\right|^{2} d x_{1} .
\end{aligned}
$$

Without loss of generality we can assume that $\lambda_{1} \leq \cdots \leq \lambda_{m}$ and let's denote $I_{j}=\left(\lambda_{j}, \lambda_{j}+c\right) \cap\left(\lambda_{j}, \lambda_{j+1}\right]$ for $1 \leq j<m, I_{m}=\left(\lambda_{m}, \lambda_{m}+c\right)$, then we have $B_{1}=\cup_{j=1}^{m} I_{j}$ and $I_{j} \cap I_{k}=\emptyset$ for $j \neq k$. By definition we have $F\left(\lambda_{j}\right)=0$ and

$$
|F(z)|^{2}=\left|\int_{\lambda_{j}}^{z} F^{\prime}(x) d x\right|^{2} \leq\left|z-\lambda_{j}\right| \int_{\lambda_{j}}^{z}\left|F^{\prime}(x)\right|^{2} d x \leq\left|z-\lambda_{j}\right| \int_{I_{j}}\left|F^{\prime}(x)\right|^{2} d x
$$

for $z \in I_{j} \subseteq\left(\lambda_{j}, \lambda_{j}+c\right)$. Thus we have

$$
\int_{I_{j}}|F(z)|^{2} d z \leq \int_{I_{j}}\left|z-\lambda_{j}\right| d z \int_{I_{j}}\left|F^{\prime}(x)\right|^{2} d x
$$

$$
\begin{aligned}
& \leq \int_{\left(\lambda_{j}, \lambda_{j}+c\right)}\left|z-\lambda_{j}\right| d z \int_{I_{j}}\left|F^{\prime}(x)\right|^{2} d x \\
& =\left(c^{2} / 2\right) \int_{I_{j}}\left|F^{\prime}(x)\right|^{2} d x
\end{aligned}
$$

Combining this with (42), we further have

$$
\begin{aligned}
& \int_{B}\left|x_{1}-x_{2}\right|\left|F\left(x_{1}\right)\right|\left|F\left(x_{2}\right)\right| d x_{1} d x_{2} \\
\leq & c^{2} \int_{B_{1}}\left|F\left(x_{1}\right)\right|^{2} d x_{1}=c^{2} \sum_{j=1}^{m} \int_{I_{j}}\left|F\left(x_{1}\right)\right|^{2} d x_{1} \\
\leq & c^{2} \sum_{j=1}^{m}\left(c^{2} / 2\right) \int_{I_{j}}\left|F^{\prime}(x)\right|^{2} d x=\left(c^{4} / 2\right) \int_{B_{1}}\left|F^{\prime}(x)\right|^{2} d x \\
\leq & \left(c^{4} / 2\right) \int_{\mathbb{R}}\left|F^{\prime}(x)\right|^{2} d x \leq\left(c^{4} / 2\right)(2 n) \int_{\mathbb{R}}|F(x)|^{2} d x=n c^{4} \int_{\mathbb{R}}|F(x)|^{2} d x
\end{aligned}
$$

which is (46). This completes the proof.
5.1. No successive small gaps. Now we can prove Lemma 8. We first need the following lemma which gives more precise meaning that there is no successive small gaps.
Lemma 10. For $A=\left(0, c_{0}\right)$ and $n>2 c_{0}^{2}+2$, we have

$$
\mathbb{P}\left(\tilde{\chi}^{(n, 2)}(A)>0\right) \leq c_{0}^{4} /(8 n)
$$

Proof. If $\widetilde{\chi}^{(n, 2)}(A)>0$, then there exist distinct $i, j, k$ such that $\lambda_{j}, \lambda_{k} \in\left(\lambda_{i}, \lambda_{i}+\right.$ $\left.c_{0} / n\right)$. Let's denote

$$
\Lambda_{j, k, c}:=\left\{\left(\lambda_{1}, \cdots, \lambda_{n}\right): \exists i \in \mathbb{Z} \cap[1, n] \text { s.t. } \lambda_{j}, \lambda_{k} \in\left(\lambda_{i}, \lambda_{i}+c\right)\right\}
$$

then we have

$$
\begin{aligned}
& \mathbb{P}\left(\widetilde{\chi}^{(n, 2)}(A)>0\right) \\
\leq & \sum_{1 \leq j<k \leq n} \mathbb{P}\left(\left(\lambda_{1}, \cdots, \lambda_{n}\right) \in \Lambda_{j, k, c_{0} / n}\right) \\
= & \mathbb{P}\left(\left(\lambda_{1}, \cdots, \lambda_{n}\right) \in \Lambda_{n-1, n, c_{0} / n}\right) n(n-1) / 2 .
\end{aligned}
$$

For fixed $\left(\lambda_{1}, \cdots, \lambda_{n-2}\right) \in \mathbb{R}^{n-2}, c>0$, as in Lemma 9 let's denote

$$
B\left(\lambda_{1}, \cdots, \lambda_{n-2}, c\right):=\cup_{i=1}^{n-2}\left(\lambda_{i}, \lambda_{i}+c\right)^{2} \subset \mathbb{R}^{2}
$$

then $\left(\lambda_{1}, \cdots, \lambda_{n}\right) \in \Lambda_{n-1, n, c}$ is equivalent to $\left(\lambda_{n-1}, \lambda_{n}\right) \in B\left(\lambda_{1}, \cdots, \lambda_{n-2}, c\right)$.
With $c=c_{0} / n>0$, we have $2 n c^{2}=2 c_{0}^{2} / n \in(0,1)$ by assumption, then by (46), we have

$$
\int_{B\left(\lambda_{1}, \cdots, \lambda_{n-2}, c_{0} / n\right)}\left|x_{1}-x_{2}\right|\left|F\left(x_{1}\right)\right|\left|F\left(x_{2}\right)\right| d x_{1} d x_{2} \leq n\left(c_{0} / n\right)^{4} \int_{\mathbb{R}}|F(x)|^{2} d x
$$

where

$$
F(x)=e^{-x^{2} / 2} \prod_{j=1}^{n-2}\left(x-\lambda_{j}\right)
$$

Hence, we have

$$
\mathbb{P}\left(\left(\lambda_{1}, \cdots, \lambda_{n}\right) \in \Lambda_{n-1, n, c_{0} / n}\right)
$$

$$
\begin{aligned}
& =\frac{1}{G_{n}} \int_{\Lambda_{n-1, n, c_{0} / n}} e^{-\sum_{i=1}^{n} \lambda_{i}^{2} / 2} \prod_{1 \leq i<j \leq n}\left|\lambda_{i}-\lambda_{j}\right| d \lambda_{1} \cdots d \lambda_{n} \\
& =\frac{1}{G_{n}} \int_{\mathbb{R}^{n-2}} d \lambda_{1} \cdots d \lambda_{n-2} e^{-\sum_{i=1}^{n-2} \lambda_{i}^{2} / 2} \prod_{1 \leq j<k \leq n-2}\left|\lambda_{j}-\lambda_{k}\right| \\
& \times \int_{B\left(\lambda_{1}, \cdots, \lambda_{n-2}, c_{0} / n\right)}\left|x_{1}-x_{2}\right| e^{-x_{1}^{2} / 2-x_{2}^{2} / 2} \prod_{i=1}^{2} \prod_{j=1}^{n-2}\left|x_{i}-\lambda_{j}\right| d x_{1} d x_{2} \\
& =\frac{1}{G_{n}} \int_{\mathbb{R}^{n-2}} d \lambda_{1} \cdots d \lambda_{n-2} e^{-\sum_{i=1}^{n-2} \lambda_{i}^{2} / 2} \prod_{1 \leq j<k \leq n-2}\left|\lambda_{j}-\lambda_{k}\right| \\
& \times \int_{B\left(\lambda_{1}, \cdots, \lambda_{n-2}, c_{0} / n\right)}\left|x_{1}-x_{2}\right|\left|F\left(x_{1}\right)\right|\left|F\left(x_{2}\right)\right| d x_{1} d x_{2} \\
& \leq \frac{n\left(c_{0} / n\right)^{4}}{G_{n}} \int_{\mathbb{R}^{n-2}} d \lambda_{1} \cdots d \lambda_{n-2} e^{-\sum_{i=1}^{n-2} \lambda_{i}^{2} / 2} \prod_{1 \leq j<k \leq n-2}\left|\lambda_{j}-\lambda_{k}\right| \\
& \times \int_{\mathbb{R}} e^{-x^{2}} \prod_{j=1}^{n-2}\left|x-\lambda_{j}\right|^{2} d x \\
& =\frac{n\left(c_{0} / n\right)^{4}}{G_{n}} G_{n-2,1}=\frac{n\left(c_{0} / n\right)^{4}}{4},
\end{aligned}
$$

where we used Lemma 1 with $k=1$ in the last step. Therefore, we have

$$
\begin{aligned}
\mathbb{P}\left(\widetilde{\chi}^{(n, 2)}(A)>0\right) & \leq \mathbb{P}\left(\left(\lambda_{1}, \cdots, \lambda_{n}\right) \in \Lambda_{n-1, n, c_{0} / n}\right) n(n-1) / 2 \\
& \leq \frac{n\left(c_{0} / n\right)^{4}}{4} n^{2} / 2=\frac{c_{0}^{4}}{8 n} .
\end{aligned}
$$

This completes the proof.
Now we can give the proof of Lemma 8 using Lemma 10 .
Proof. Let $c_{0}$ be such that $A \subset\left(0, c_{0}\right)$ and $A_{1}=\left(0, c_{0}\right)$. Then $\chi^{(n)}(A)-\widetilde{\chi}^{(n)}(A) \neq$ 0 implies $\widetilde{\chi}^{(n, j)}(A)>0$ for some $j>1$ and thus we must have $\widetilde{\chi}^{(n, 2)}\left(A_{1}\right) \geq$ $\widetilde{\chi}^{(n, j)}\left(A_{1}\right) \geq \widetilde{\chi}^{(n, j)}(A)>0$. For $n>2 c_{0}^{2}+2$, by Lemma 10 we deduce that

$$
\mathbb{P}\left(\chi^{(n)}(A)-\widetilde{\chi}^{(n)}(A) \neq 0\right) \leq \mathbb{P}\left(\widetilde{\chi}^{(n, 2)}\left(A_{1}\right)>0\right) \leq c_{0}^{4} /(8 n) \rightarrow 0
$$

which completes the proof.

## 6. Integral inequalities of Two-component log-Gas

In this section, we will prove several useful inequalities regarding the two-component log-gas, which is one of the crucial steps in proving the convergence of the factorial moments of $\widetilde{\chi}^{(n)}$ (see Lemma 12).

Let $A=\left(0, c_{0}\right), n>2 k$, by the definition of $\rho^{(k, n)}$, we have

$$
\begin{equation*}
\mathbb{E} \rho^{(k, n)}\left(A^{k}\right)=\frac{n!}{(n-2 k)!2^{k} G_{n}} \int_{\Sigma_{n, k, c_{0} / n}}\left|J_{n}\left(\lambda_{1}, \cdots, \lambda_{n}\right)\right| d \lambda_{1} \cdots d \lambda_{n} \tag{51}
\end{equation*}
$$

where $J_{n}$ is defined in (15) and

$$
\begin{equation*}
\Sigma_{n, k, c}=\left\{\left(\lambda_{1}, \cdots, \lambda_{n}\right) \in \mathbb{R}^{n}:\left|\lambda_{j}-\lambda_{j-k}\right|<c, \forall n-k<j \leq n\right\} \tag{52}
\end{equation*}
$$

i.e., $\Sigma_{n, k, c}$ is the set $\left(\lambda_{1}, \cdots, \lambda_{n}\right) \in \mathbb{R}^{n}$ with $k$ pairs $\left(\lambda_{j}, \lambda_{j-k}\right)$ such that $\mid \lambda_{j}-$ $\lambda_{j-k} \mid<c$.

We will first prove the inequality (55) below regarding the two-component loggas. The significance of such type inequality is that it will imply the bounds between the integration of the joint density over the set $\Sigma_{n, k, c_{0} / n}$, i.e., $\mathbb{E} \rho^{(k, n)}\left(A^{k}\right)$ and the partition function $G_{n-2 k, k}$ of the two-component log-gas which consists of $n-2 k$ particles with charge $q=1$ and $k$ particles with charge $q=2$ (see Lemma 11).

For $0 \leq l \leq k$, let's denote the following integral of the two-component log-gas

$$
E_{n, k, l}(c):=\left.\int_{\Sigma_{n-l, k-l, c}} d \lambda_{1} \cdots d \lambda_{n-l} e^{-\sum_{i=1}^{n-l} q_{i} \lambda_{i}^{2} / 2} \prod_{j<m}\left|\lambda_{j}-\lambda_{m}\right|^{q_{j} q_{m}}\right|_{q_{s}=1+\chi_{\{s \leq l\}}},
$$

where $\Sigma_{n-l, k-l, c}$ is defined via (52). By definition of $G_{n_{1}, n_{2}}$ (recall (24)), we first have

$$
E_{n, k, k}(c)=G_{n-2 k, k}
$$

We also have

$$
\begin{equation*}
E_{n, k, 0}(c)=\int_{\Sigma_{n, k, c}}\left|J_{n}\left(\lambda_{1}, \cdots, \lambda_{n}\right)\right| d \lambda_{1} \cdots d \lambda_{n} \tag{53}
\end{equation*}
$$

which implies

$$
\begin{equation*}
\mathbb{E} \rho^{(k, n)}\left(A^{k}\right)=\frac{n!}{(n-2 k)!2^{k} G_{n}} E_{n, k, 0}\left(c_{0} / n\right) \tag{54}
\end{equation*}
$$

We will show that (for $0<2 n c^{2}<1$ )

$$
\begin{equation*}
\left(1-n c^{2}\right) c^{2} \leq \frac{E_{n, k, l-1}(c)}{E_{n, k, l}(c)} \leq c^{2} \tag{55}
\end{equation*}
$$

In fact, after changing the order of variables, we can rewrite

$$
\begin{aligned}
& E_{n, k, l-1}(c)=\int_{\Sigma_{n-l-1, k-l, c}} d \lambda_{1} \cdots d \lambda_{n-l-1} e^{-\sum_{i=1}^{n-l-1} q_{i} \lambda_{i}^{2} / 2} \prod_{1 \leq j<m \leq n-l-1}\left|\lambda_{j}-\lambda_{m}\right|^{q_{j} q_{m}} \\
& \times\left.\int_{\mathbb{R}} d x_{1} \int_{x_{1}-c}^{x_{1}+c} d x_{2}\left|x_{1}-x_{2}\right| e^{-x_{1}^{2} / 2-x_{2}^{2} / 2} \prod_{j=1}^{2} \prod_{m=1}^{n-l-1}\left|x_{j}-\lambda_{m}\right|^{q_{m}}\right|_{q_{s}=1+\chi_{\{s \leq l-1\}}},
\end{aligned}
$$

and

$$
\begin{aligned}
& E_{n, k, l}(c)=\int_{\Sigma_{n-l-1, k-l, c}} d \lambda_{1} \cdots d \lambda_{n-l-1} e^{-\sum_{i=1}^{n-l-1} q_{i} \lambda_{i}^{2} / 2} \prod_{1 \leq j<m \leq n-l-1}\left|\lambda_{j}-\lambda_{m}\right|^{q_{j} q_{m}} \\
& \times\left.\int_{\mathbb{R}} d x_{1} e^{-x_{1}^{2}} \prod_{m=1}^{n-l-1}\left|x_{1}-\lambda_{m}\right|^{2 q_{m}}\right|_{q_{s}=1+\chi_{\{s \leq l-1\}}}
\end{aligned}
$$

Then (55) follows from (43) by taking

$$
\begin{equation*}
F(x)=e^{-x^{2} / 2} \prod_{j=1}^{n-l-1}\left|x-\lambda_{m}\right|^{q_{m}} \tag{56}
\end{equation*}
$$

By (55) we will have

$$
\begin{equation*}
E_{n, k, l}(c) \leq\left(c^{2}\right)^{k-l} E_{n, k, k}(c)=c^{2(k-l)} G_{n-2 k, k} \tag{57}
\end{equation*}
$$

For $n>2 k$, given any interval $A$, let's denote

$$
\begin{equation*}
\Sigma_{n, k, A}=\left\{\left(\lambda_{1}, \cdots, \lambda_{n}\right) \in \mathbb{R}^{n}:\left|\lambda_{j}-\lambda_{j-k}\right| \in A, \forall n-k<j \leq n\right\} \tag{58}
\end{equation*}
$$

For $0 \leq l \leq k$, let's denote

$$
E_{n, k, l}(A):=\int_{\Sigma_{n-l, k-l, A}} d \lambda_{1} \cdots d \lambda_{n-l} e^{-\sum_{i=1}^{n-l} q_{i} \lambda_{i}^{2} / 2} \prod_{j<m}\left|\lambda_{j}-\lambda_{m}\right|^{q_{j} q_{m}}
$$

where $q_{s}=1+\chi_{\{0<s \leq l\}}$ and $\Sigma_{n-l, k-l, A}$ is defined via (58). Then we have

$$
\begin{equation*}
E_{n, k, 0}(A)=\int_{\Sigma_{n, k, A}}\left|J_{n}\left(\lambda_{1}, \cdots, \lambda_{n}\right)\right| d \lambda_{1} \cdots d \lambda_{n} \tag{59}
\end{equation*}
$$

and

$$
\begin{equation*}
E_{n, k, k}(A)=G_{n-2 k, k} \tag{60}
\end{equation*}
$$

With such notations, as before, we have

$$
\begin{align*}
\mathbb{E} \rho^{(k, n)}\left(A^{k}\right) & =\frac{n!}{(n-2 k)!2^{k} G_{n}} \int_{\Sigma_{n, k, A / n}}\left|J_{n}\left(\lambda_{1}, \cdots, \lambda_{n}\right)\right| d \lambda_{1} \cdots d \lambda_{n}  \tag{61}\\
& =\frac{n!}{(n-2 k)!2^{k}} \frac{E_{n, k, 0}(A / n)}{G_{n}}
\end{align*}
$$

We also need inequalities similar to (55).
Lemma 11. If $A \subset\left(0, c_{1}\right), 2 n c_{1}^{2} \in(0,1), n>2 k, n, k$ are positive integers, then we have

$$
\left(1-n c_{1}^{2}\right)^{k}\left(2 \int_{A} u d u\right)^{k} G_{n-2 k, k} \leq E_{n, k, 0}(A) \leq\left(2 \int_{A} u d u\right)^{k} G_{n-2 k, k}
$$

Proof. Let $A_{1}=A \cup(-A)$, after changing the order of variables, we can rewrite

$$
\begin{aligned}
& E_{n, k, l-1}(A)=\int_{\Sigma_{n-l-1, k-l, A}} d \lambda_{1} \cdots d \lambda_{n-l-1} e^{-\sum_{i=1}^{n-l} q_{i} \lambda_{i}^{2} / 2} \prod_{1 \leq j<m \leq n-l-1}\left|\lambda_{j}-\lambda_{m}\right|^{q_{j} q_{m}} \\
& \times\left.\int_{\mathbb{R}} d x_{1} \int_{x_{1}+A_{1}} d x_{2}\left|x_{1}-x_{2}\right| e^{-x_{1}^{2} / 2-x_{2}^{2} / 2} \prod_{j=1}^{2} \prod_{m=1}^{n-l-1}\left|x_{j}-\lambda_{m}\right|^{q_{m}}\right|_{q_{s}=1+x_{\{s \leq l-1\}}}
\end{aligned}
$$

and

$$
\begin{aligned}
& E_{n, k, l}(A)=\int_{\Sigma_{n-l-1, k-l, A}} d \lambda_{1} \cdots d \lambda_{n-l-1} \prod_{1 \leq j<m \leq n-l-1}\left|\lambda_{j}-\lambda_{m}\right|^{q_{j} q_{m}} \\
& \times\left.\int_{\mathbb{R}} d x_{1} e^{-x_{1}^{2}} \prod_{m=1}^{n-l-1}\left|x_{1}-\lambda_{m}\right|^{2 q_{m}}\right|_{q_{s}=1+\chi_{\{s \leq l-1\}}}
\end{aligned}
$$

Taking $F(x)$ as in (56) again, by (45) we have

$$
\left(1-n c_{1}^{2}\right) \cdot 2 \int_{A} u d u \leq \frac{E_{n, k, l-1}(A)}{E_{n, k, l}(A)} \leq 2 \int_{A} u d u
$$

and the result follows by induction and (60).

## 7. Proof of Theorem 1

By Lemma 8 and the moment method, Theorem will be proved if we can prove the following convergence of the factorial moment

$$
\begin{equation*}
\lim _{n \rightarrow+\infty} \mathbb{E}\left(\frac{\left(\tilde{\chi}^{(n)}(A)\right)!}{\left(\widetilde{\chi}^{(n)}(A)-k\right)!}\right)=\left(\frac{1}{4} \int_{A} u d u\right)^{k} \tag{62}
\end{equation*}
$$

for every positive integer $k$ and bounded interval $A \subset \mathbb{R}_{+}$. Actually, combining Lemma (1) (62) is equivalent to

Lemma 12. For any bounded interval $A \subset \mathbb{R}_{+}$and any positive integer $k \geq 1$, we have

$$
\mathbb{E}\left(\frac{\left(\tilde{\chi}^{(n)}(A)\right)!}{\left(\widetilde{\chi}^{(n)}(A)-k\right)!}\right)-\left(\int_{A} u d u\right)^{k} \frac{G_{n-2 k, k}}{G_{n}} \rightarrow 0
$$

as $n \rightarrow+\infty$.
We will first use Lemma 7 to prove that

$$
\begin{equation*}
\lim _{n \rightarrow+\infty}\left(\mathbb{E} \frac{\left(\widetilde{\chi}^{(n)}(A)\right)!}{\left(\widetilde{\chi}^{(n)}(A)-k\right)!}-\mathbb{E} \rho^{(k, n)}\left(A^{k}\right)\right)=0 \tag{63}
\end{equation*}
$$

and then use Lemma 11 to prove that

$$
\begin{equation*}
\lim _{n \rightarrow+\infty}\left(\mathbb{E}\left(\rho^{(k, n)}\left(A^{k}\right)\right)-\left(\int_{A} u d u\right)^{k} \frac{G_{n-2 k, k}}{G_{n}}\right)=0 \tag{64}
\end{equation*}
$$

then Lemma 12 follows from (63) and (64), and hence we complete the proof of Theorem 1

For the rest of the article, for any bounded interval $A \subset \mathbb{R}_{+}$, let $c_{1}$ be such that $A \subset\left(0, c_{1}\right)$, and $A_{1}=\left(0,2 c_{1}\right)$, then $A \subset A_{1}$; let's denote $c_{n}=c_{1} / n$, then $2 n\left(2 c_{n}\right)^{2}=8 n^{-1} c_{1}^{2} \in(0,1)$ for $n$ large enough. By (54), (57) with $l=0$ and Lemma 1] we have

$$
\begin{align*}
& \mathbb{E} \rho^{(k, n)}\left(A_{1}^{k}\right)=\frac{n!}{(n-2 k)!2^{k}} \frac{E_{n, k, 0}\left(2 c_{n}\right)}{G_{n}}  \tag{65}\\
\leq & \frac{n!}{(n-2 k)!2^{k}} \frac{G_{n-2 k, k}}{G_{n}}\left(2 c_{n}\right)^{2 k} \leq \frac{n^{2 k}}{2^{k}} 2^{-2 k}\left(\frac{2 c_{1}}{n}\right)^{2 k}=2^{-k} c_{1}^{2 k} .
\end{align*}
$$

Let $a$ be defined in Lemma 7 then we have

$$
\rho^{(k, n)}\left(A_{1}^{k}\right) \geq \frac{(a+1)!}{(a+1-2 k)!2^{k}} \geq \frac{(a+1-2 k)_{+}^{2 k}}{2^{k}}
$$

and hence

$$
\mathbb{E}(a+1-2 k)_{+}^{2 k} \leq 2^{k} \mathbb{E} \rho^{(k, n)}\left(A_{1}^{k}\right) \leq c_{1}^{2 k}
$$

here we denote $f_{+}:=\max (f, 0)$. Since $a, k \in \mathbb{Z}, a \geq 0, k \geq 1$, by Hölder's inequality we have

$$
\mathbb{E}(a+1-2 k)_{+}^{k} \leq\left(\mathbb{E}(a+1-2 k)_{+}^{2 k}\right)^{\frac{1}{2}}(\mathbb{P}(a \geq 2))^{\frac{1}{2}} \leq c_{1}^{k}(\mathbb{P}(a \geq 2))^{\frac{1}{2}}
$$

Moreover, it's easy to check

$$
(a-1)_{+} \leq \max \left(2(a+1-2 k)_{+},(4 k-4) \chi_{a \geq 2}\right)
$$

and thus

$$
(a-1)_{+}^{k} \leq 2^{k}(a+1-2 k)_{+}^{k}+(4 k-4)^{k} \chi_{a \geq 2}
$$

hence, we have

$$
\begin{aligned}
\mathbb{E}(a-1)_{+}^{k} & \leq 2^{k} \mathbb{E}(a+1-2 k)_{+}^{k}+(4 k-4)^{k} \mathbb{P}(a \geq 2) \\
& \leq 2^{k} c_{1}^{k}(\mathbb{P}(a \geq 2))^{\frac{1}{2}}+(4 k-4)^{k} \mathbb{P}(a \geq 2)
\end{aligned}
$$

On the other hand, $a \geq 2$ is equivalent to $\widetilde{\chi}^{(n, 2)}\left(A_{1}\right)>0$, by Lemma 10 we have

$$
\mathbb{P}(a \geq 2)=\mathbb{P}\left(\tilde{\chi}^{(n, 2)}\left(A_{1}\right)>0\right) \leq 2 c_{1}^{4} / n \rightarrow 0
$$

and thus we further have

$$
\begin{equation*}
\lim _{n \rightarrow+\infty} \mathbb{E}(a-1)_{+}^{k}=0 \tag{66}
\end{equation*}
$$

By (38) in Lemma 7 we have

$$
\left(\tilde{\chi}^{(n)}(A)\right)^{k} \leq 2 \rho^{(k, n)}\left(A^{k}\right) \text { or }\left(\tilde{\chi}^{(n)}(A)\right)^{k} \leq 2 k(k-1) a\left(\tilde{\chi}^{(n)}(A)\right)^{k-1}
$$

therefore,

$$
\left(\widetilde{\chi}^{(n)}(A)\right)^{k} \leq \max \left(2 \rho^{(k, n)}\left(A^{k}\right),(2 k(k-1) a)^{k}\right)
$$

and thus we have

$$
\mathbb{E}\left(\tilde{\chi}^{(n)}(A)\right)^{k} \leq 2 \mathbb{E}\left(\rho^{(k, n)}\left(A^{k}\right)\right)+(2 k(k-1))^{k} \mathbb{E}\left(a^{k}\right)
$$

By (65), (66) and the fact that $\mathbb{E} \rho^{(k, n)}\left(A^{k}\right) \leq \mathbb{E} \rho^{(k, n)}\left(A_{1}^{k}\right)$ since $A \subset A_{1}$, we further have

$$
\begin{equation*}
\limsup _{n \rightarrow+\infty} \mathbb{E}\left(\tilde{\chi}^{(n)}(A)\right)^{k}<+\infty \tag{67}
\end{equation*}
$$

Note that (63) is clearly true for $k=1$ by definitions. For $k \geq 2$, by (37) in Lemma 7. Hölder's inequality, (66) and (67), we have

$$
\begin{aligned}
0 & \leq \mathbb{E}\left(\frac{\left(\widetilde{\chi}^{(n)}(A)\right)!}{\left(\widetilde{\chi}^{(n)}(A)-k\right)!}-\rho^{(k, n)}\left(A^{k}\right)\right) \\
& \leq k(k-1) \mathbb{E}\left((a-1)_{+}\left(\widetilde{\chi}^{(n)}(A)\right)^{k-1}\right) \\
& \leq k(k-1)\left(\mathbb{E}\left((a-1)_{+}^{k}\right)\right)^{1 / k}\left(\mathbb{E}\left(\widetilde{\chi}^{(n)}(A)^{k}\right)\right)^{1-1 / k} \rightarrow 0
\end{aligned}
$$

as $n \rightarrow+\infty$, which finishes the proof of (63).
Now we prove (64). By (61) and changing of variables, we have

$$
\begin{aligned}
& \mathbb{E}\left(\rho^{(k, n)}(A)\right)-\left(\int_{A} u d u\right)^{k} \frac{G_{n-2 k, k}}{G_{n}} \\
= & \frac{n!}{(n-2 k)!2^{k}} \frac{E_{n, k, 0}(A / n)}{G_{n}}-\left(\int_{A / n} u d u\right)^{k} \frac{n^{2 k} G_{n-2 k, k}}{G_{n}} \\
= & \frac{n^{2 k}}{2^{k} G_{n}}\left(E_{n, k, 0}(A / n)-\left(2 \int_{A / n} u d u\right)^{k} G_{n-2 k, k}\right) \\
& -\left(n^{2 k}-\frac{n!}{(n-2 k)!}\right) \frac{E_{n, k, 0}(A / n)}{2^{k} G_{n}}
\end{aligned}
$$

We first notice that

$$
\begin{aligned}
0 & \leq n^{2 k}-\frac{n!}{(n-2 k)!}=n^{2 k}-\prod_{j=0}^{2 k-1}(n-j)=n^{2 k}-n^{2 k} \prod_{j=0}^{2 k-1}(1-j / n) \\
& \leq n^{2 k}-n^{2 k}\left(1-\sum_{j=0}^{2 k-1} j / n\right)=n^{2 k} \sum_{j=0}^{2 k-1} j / n=n^{2 k-1} k(2 k-1)
\end{aligned}
$$

We also have $A / n \subset\left(0, c_{1} / n\right)$ and $2 n\left(c_{1} / n\right)^{2} \in(0,1)$ for $n$ large enough, then by (53), (57) and (59), we have

$$
0 \leq E_{n, k, 0}(A / n) \leq E_{n, k, 0}\left(c_{1} / n\right) \leq G_{n-2 k, k}\left(c_{1} / n\right)^{2 k}
$$

Therefore, using Lemma we have

$$
\begin{aligned}
0 & \leq\left(n^{2 k}-\frac{n!}{(n-2 k)!}\right) \frac{E_{n, k, 0}(A / n)}{2^{k} G_{n}} \\
& \leq n^{2 k-1} k(2 k-1) \frac{G_{n-2 k, k}}{2^{k} G_{n}}\left(c_{1} / n\right)^{2 k} \\
& =n^{-1} k(2 k-1) 2^{-3 k} c_{1}^{2 k}
\end{aligned}
$$

By Lemma 1 and Lemma 11] we have

$$
\begin{aligned}
& \frac{n^{2 k}}{2^{k} G_{n}}\left|E_{n, k, 0}(A / n)-\left(2 \int_{A / n} u d u\right)^{k} G_{n-2 k, k}\right| \\
\leq & \frac{n^{2 k}}{2^{k} G_{n}}\left(1-\left(1-n\left(c_{1} / n\right)^{2}\right)^{k}\right)\left(2 \int_{A / n} u d u\right)^{k} G_{n-2 k, k} \\
\leq & \frac{n^{2 k}}{2^{k} G_{n}}\left(k n\left(c_{1} / n\right)^{2}\right)\left(2 \int_{0}^{c_{1} / n} u d u\right)^{k} G_{n-2 k, k} \\
= & \frac{n^{2 k}}{2^{k} G_{n}}\left(k c_{1}^{2} / n\right)\left(c_{1} / n\right)^{2 k} G_{n-2 k, k} \\
= & \frac{G_{n-2 k, k}}{2^{k} G_{n}}\left(k c_{1}^{2 k+2} / n\right)=\frac{k c_{1}^{2 k+2}}{2^{3 k} n} .
\end{aligned}
$$

Therefore, we have

$$
\left|\mathbb{E}\left(\rho^{(k, n)}(A)\right)-\left(\int_{A} u d u\right)^{k} \frac{G_{n-2 k, k}}{G_{n}}\right| \leq \frac{k c_{1}^{2 k+2}+k(2 k-1) c_{1}^{2 k}}{2^{3 k} n}
$$

which implies (64). Therefore, we finish the proof of Lemma 12 and thus the whole proof of Theorem 1 .

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Beijing International Center for Mathematical Research, Peking University, Beijing, China, 100871.

E-mail address: renjie@math.pku.edu.cn
E-mail address: gtian@math.pku.edu.cn
E-mail address: jnwdyi@pku.edu.cn


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