K-stability and Kähler-Einstein metrics

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Contents

1 Introduction

In this paper, we solve a folklore conjecture [1] on Fano manifolds without non-trivial holomorphic vector fields. The main technical ingredient is a conic version of Cheeger-Colding-Tian's theory on compactness of Kähler-Einstein manifolds. This enables us to prove the partial C^0-estimate for conic Kähler-Einstein metrics.

A Fano manifold is a projective manifold with positive first Chern class $c_1(M)$. Its holomorphic fields form a Lie algebra $\eta(M)$. The folklore conjecture states: *If $\eta(M) = \{0\}$, then M admits a Kähler-Einstein metric if and only if M is K-stable with respect to the anti-canonical bundle K_M^{-1}.* Its necessary part was established in [Ti97]. The following gives the sufficient part of this conjecture.

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[1] It is often referred as the Yau-Tian-Donaldson conjecture

Theorem 1.1. *Let M be a Fano manifold canonically polarized by the anti-canonical bundle K_M^{-1}. If M is K-stable, then it admits a Kähler-Einstein metric.*

An older approach for proving this theorem is to solve the following complex Monge-Ampere equations by the continuity method:

$$(\omega + \sqrt{-1}\,\partial\bar{\partial}\varphi)^n \,=\, e^{h-t\varphi}\omega^n, \quad \omega + \sqrt{-1}\,\partial\bar{\partial}\varphi \,>\, 0, \tag{1.1}$$

where ω is a given Kähler metric with its Kähler class $[\omega] = 2\pi c_1(M)$ and h is uniquely determined by

$$\mathrm{Ric}(\omega) - \omega \,=\, \sqrt{-1}\,\partial\bar{\partial}h, \qquad \int_M (e^h - 1)\,\omega^n \,=\, 0.$$

Let I be the set of t for which (1.1) is solvable. Then we have known: (1) By the well-known Calabi-Yau theorem, I is non-empty; (2) In 1983, Aubin proved that I is open [Au83]; (3) If we can have an a priori C^0-estimate for the solutions of (1.1), then I is closed and consequently, there is a Kähler-Einstein metric on M.

However, the C^0-estimate does not hold in general since there are many Fano manifolds which do not admit any Kähler-Einstein metrics. The existence of Kähler-Einstein metrics required certain geometric stability on the underlying Fano manifolds. In early 90's, I proposed a program towards establishing the existence of Kähler-Einstein metrics. The key technical ingredient of this program is a conjectured partial C^0-estimate. If we can affirm this conjecture for the solutions of (1.1), then we can use the K-stability to derive the a prior C^0-estimate and the Kähler-Einstein metric. The K-stability was first introduced in [Ti97] as a test for the properness of the K-energy restricted to a finite dimensional family of Kähler metrics induced by a fixed embedding by pluri-anti-canonical sections.[2] However, such a conjecture on partial C^0-estimates is still open except for Kähler-Einstein metrics.

In [Do10], in his approach to solving the above folklore conjecture through the b-stability, Donaldson suggested a continuity method by deforming through conic Kähler-Einstein metrics. Those are metrics with cone angle along a divisor. For simplicity, here we consider only the case of smooth divisors.

Let M be a compact Kähler manifold and $D \subset M$ be a smooth divisor. A conic Kähler metric on M with angle $2\pi\beta$ ($0 < \beta \leq 1$) along D is a Kähler metric on $M\backslash D$ that is asymptotically equivalent along D to the model conic metric

$$\omega_{0,\beta} \,=\, \sqrt{-1}\left(\frac{dz_1 \wedge d\bar{z}_1}{|z_1|^{2-2\beta}} \,+\, \sum_{j=2}^n dz_j \wedge d\bar{z}_j\right),$$

where $z_1, z_2, \cdots, z_n$ are holomorphic coordinates such that $D = \{z_1 = 0\}$ locally. Each conic Kähler metric can be given by its Kähler form ω which represents

[2] The K-stability was reformulated in more algebraic ways (see [Do02], [Pa12] and [Ti13]).

a cohomology class $[\omega]$ in $H^{1,1}(M,\mathbb{C}) \cap H^2(M,\mathbb{R})$, referred as the Kähler class. A conic Kähler-Einstein metric is a conic Kähler metric which is also Einstein outside conic points.

In this paper, we only need to consider the following conic Kähler-Einstein metrics: Let M be a Fano manifold and D be a smooth divisor which represents the Poincaré dual of $\lambda c_1(M)$. We call ω a conic Kähler-Einstein with cone angle $2\pi\beta$ along D if it has $2\pi c_1(M)$ as its Kähler class and satisfies

$$\mathrm{Ric}(\omega) \,=\, \mu\,\omega \,+\, 2\pi(1-\beta)\,[D]. \tag{1.2}$$

Here the equation on M is in the sense of currents, while it is classical outside D. We will require $\mu > 0$ which is equivalent to $(1-\beta)\lambda < 1$. As in the smooth case, each conic Kähler metric ω with $[\omega] = 2\pi c_1(M)$ is the curvature of a Hermitian metric $||\cdot||$ on the anti-canonical bundle K_M^{-1}. The difference is that the Hermitian metric here is not smooth, but it is Hölder continuous.

Donaldson's continuity method was originally proposed as follows: Assume that $\lambda = 1$, i.e., D be a smooth anti-canonical divisor. It follows from [TY90] that there is a complete Calabi-Yau metric on $M\backslash D$. It was conjectured that this complete metric is the limit of Kähler-Einstein metrics with cone angle $2\pi\beta \mapsto 0$. If this is true, then the set E of $\beta \in (0,1]$ such that there is a conic Kähler metric satisfying (1.2) is non-empty. It is proved in [Do10] that E is open. Then we are led to proving that E is closed.

A problem with Donaldson's original approach arose because we do not know if a Fano manifold M always has a smooth anti-canonical divisor D. Possibly, there are Fano manifolds which do not admit smooth anti-canonical divisors. At least, it seems to be a highly non-trivial problem whether or not any Fano manifold admits a smooth anti-canonical divisor. Fortunately, Li and Sun bypassed this problem. Inspired by [JMR11], they modified Donaldson's original approach by allowing $\lambda > 1$. They observed that the main existence theorem in [JMR11], coupled with an estimate on log-α invariants in [Be13], implies the existence of conic Kähler-Einstein metrics with cone angle $2\pi\beta$ so long as $\mu = 1-(1-\beta)\lambda$ is sufficiently small. Now we define E to be set of $\beta \in (1-\lambda^{-1}, 1]$ such that there is a conic Kähler metric satisfying (1.2). Then E is non-empty. It follows from [Do10] that E is open. The difficult part is to prove that E is closed.

The construction of Kähler-Einstein metrics with cone angle $2\pi\beta$ can be reduced to solving complex Monge-Ampere equations:

$$(\omega_\beta \,+\, \sqrt{-1}\partial\bar\partial\varphi)^n \,=\, e^{h_\beta - \mu\varphi}\,\omega_\beta^n, \tag{1.3}$$

where ω_β is a suitable family of conic Kähler metrics with $[\omega_\beta] = 2\pi c_1(M)$ and cone angle $2\pi\beta$ along D and h_β is determined by

$$\mathrm{Ric}(\omega_\beta) \,=\, \mu\,\omega_\beta \,+\, 2\pi(1-\beta)\,[D] + \sqrt{-1}\,\partial\bar\partial h_\beta \quad \text{and} \quad \int_M (e^{h_\beta}-1)\,\omega_\beta^n \,=\, 0.$$

As shown in [JMR11], it is crucial for solving (1.3) to establish an a priori C^0-estimate for its solutions. Such a C^0-estimate does not hold in general.

Therefore, following my program on the existence of Kähler-Einstein metrics through the Aubin's continuity method, we can first establish a partial C^0-estimate and then use the K-stability to conclude the C^0-estimate, consequently, the existence of Kähler-Einstein metrics on Fano manifolds which are K-stable.

For any integer $\lambda > 0$ and $\beta > 0$, let $\mathcal{E}(\lambda, \beta)$ be the set of all triples (M, D, ω), where M is a Fano manifold, D is a smooth divisor whose Poincare dual is $\lambda\, c_1(M)$ and ω is a conic Kähler-Einstein metric on M with cone angle $2\pi\beta$ along D. For any $\omega \in \mathcal{E}(\lambda, \beta)$, choose a Hölder continuous Hermitian metric h with ω as its curvature form, then we have an induced inner product $< \cdot, \cdot >$ on each $H^0(M, K_M^{-\ell})$ as follows:

$$< S, S' > = \int_M h^\ell(S, S')\, \omega^n, \qquad \forall\, S, S' \in H^0(M, K_M^{-\ell}). \tag{1.4}$$

Let $\{S_i\}_{0\le i\le N}$ be any orthonormal basis of $H^0(M, K_M^{-\ell})$ with respect to this induced inner product by h and ω, then, as done in the smooth case, we can introduce a function

$$\rho_{\omega,\ell}(x) \;=\; \sum_{i=0}^N ||S_i||_h^2(x). \tag{1.5}$$

The following provides the partial C^0-estimate for conic Kähler-Einstein metrics. The estimate is needed in completing the proof of Theorem 1.1.

Theorem 1.2. *For any fixed λ and $\beta_0 > 1 - \lambda^{-1}$, there are uniform constants $c_k = c(k, n, \lambda, \beta_0) > 0$ for $k \ge 1$ and $\ell_i \to \infty$ such that for any $\beta \ge \beta_0$ and $\omega \in \mathcal{E}(\lambda, \beta)$, we have for $\ell = \ell_i$,*

$$\rho_{\omega,\ell} \;\ge\; c_\ell \;> 0. \tag{1.6}$$

We expect that this theorem holds for more general conic Kähler metrics. In fact, our method for proving the above theorem should be also applicable to establishing the partial C^0-estimate for conic Kähler-Einstein metrics in more general cases.

A crucial tool in proving Theorem 1.2 is an extension of Cheeger-Colding-Tian's compactness theorem for Kähler-Einstein metrics to the conic cases.

Theorem 1.3. *Let M be a Fano manifold with a smooth pluri-anti-canonical divisor D of $K_M^{-\lambda}$. Assume that ω_i be a sequence of conic Kähler-Einstein metrics with cone angle $2\pi\beta_i$ along D satisfying:*

$$\mathrm{Ric}(\omega_i) \;=\; \mu_i\, \omega_i \;+\; 2\pi(1-\beta_i)\,[D], \qquad \mu_i \;=\; 1 - (1-\beta_i)\lambda.$$

where $\mu_i = 1 - (1-\beta_i)\lambda > 0$. We further assume that $\lim \mu_i = \mu_\infty > 0$ and (M, ω_i) converge to a length space (M_∞, d_∞) in the Gromov-Hausdorff topology. Then there is a closed subset $\bar{\mathcal{S}} \cup D_\infty$ of M_∞, where $\bar{\mathcal{S}}$ is of codimension at least 4 and D_∞ is the limit of D in the Gromov-Hausdorff topology, such that M_∞ is a smooth Kähler manifold and d_∞ is induced by a smooth Kähler-Einstein *metric outside $\bar{\mathcal{S}} \cup D_\infty \subset M_\infty$. Furthermore, (M, ω_i) converge to $(M_\infty, \omega_\infty)$ outside $\bar{\mathcal{S}} \cup D_\infty$ in the C^∞-topology.*[3]

Extra technical inputs are needed in order to establish such an extension.

The organization of this paper is as follows: In the next section, we prove an approximation theorem which states any conic Kähler-Einstein metric can be approximated by smooth Kähler metrics with the same lower bound on Ricci curvature. In section 3, we give an extension of my works with Cheeger-Colding in [CCT02] to conic Kähler-Einstein manifolds. In section 4, we prove the smooth convergence for conic Kähler-Einstein metrics. In the smooth case, it is based on a result of M. Anderson. However, the arguments do not apply to the conic case. We have to introduce a new method. In Section 5, we prove Theorem 1.2, i.e., the partial C^0-estimate for conic Kähler-Einstein metrics. In Section 6, we prove Theorem 1.1. We provide two proofs. One is conceptually better and works for more general cases, while the other is simpler and works for the case of Kähler-Einstein metrics on Fano manifolds. In Appendix 1, we give a detailed proof for a technical lemma in Section 5, i.e., Lemma 5.8. In Appendix 2, for the readers' convenience, we outline a proof of a previous theorem due to B. Wang and myself. This theorem will be used in Section 3 and 4 when we prove an extension of [CCT02] when the cone angles tend to 1.

The existence of Kähler-Einstein metrics on K-stable Fano manifold was first mentioned in my talk during the conference "Conformal and Kähler Geometry" held at IHP in Paris from September 17 to September 21 of 2012. On October 25 of 2012, in my talk at the Blainefest held at Stony Brook University, I outlined my proof of Theorem 1.1, particularly, I described how to extend [CCT02] to conic Kähler-Einstein manifolds[4], including a sharp approximation theorem for conic Kähler-Einstein metrics by smooth metrics with Ricci curvature bounded from below and key ingredient in proving the smooth convergence when cone angles tend to 1. I mentioned that the partial C^0-estimate in the conic case can be proved by using the extension of Cheeger-Colding-Tian compactness and the arguments from [DS14] and also [Ti13]. I also mentioned that the K-stability is equivalent to the properness of the K-energy restricted to the family of Bergmann type metrics and the partial C^0-estimate reduces the required C^0-estimate to this properness. On October 30 of 2012, X.X. Chen, S. Donaldson and S. Sun posted on the arXiv a short note [CDS14] in which they also announced a proof of Theorem 1.1 and gave an outline of the proof. On November 20 of 2012, I posted on the arXiv the first version of this paper which contains all necessary results for proving Theorem 1.1, and on January 28 of 2013, the second version of my paper which contains a proof of Theorem 1.1. After their announcement, Chen-Donaldson-Sun posted on the arXiv three papers [CDS15, I, II, III] on November 19 of 2012, December 19 of 2012 and February 1 of 2013 in which they also presented a proof of Theorem 1.1.

[3] Actually, we will prove (see Theorem 5.9) that M_∞ is a normal variety embedded in some $\mathbb{C}P^N$ and $\bar{\mathcal{S}}$ is a subvariety of complex codimension at least 2.

[4] My work with Cheeger and Colding [CCT02] is definitely needed in establishing the partial C^0-estimate which is crucial for proving Theorem 1.1.

In this new version, after feedbacks from referees and others, I improved the presentation of this paper and provided additional details.

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2 Smoothing conic Kähler-Einstein metrics

In this section, we address the question: *Can one approximate a conic Kähler-Einstein metrics by smooth Kähler metrics with Ricci curvature bounded from below*? For the sake of this paper, we confine ourselves to the case of positive scalar curvature. Our approach can be adapted to other cases where the scalar curvature is non-positive. In fact, the proof is even simpler.

Let ω be a conic Kähler-Einstein metric on M with cone angle $2\pi\beta$ along D, where D is a smooth divisor whose Poincaré dual is equal to $\lambda\, c_1(M)$, in particular, ω satisfies (1.2) for $\mu = 1-(1-\beta)\lambda > 0$. For any smooth Kähler metric ω_0 with $[\omega_0] = 2\pi c_1(M)$, we can write $\omega = \omega_0 + \sqrt{-1}\,\partial\bar\partial\varphi$ for some smooth function φ on $M\backslash D$. Note that φ is Hölder continuous on M. Define h_0 by

$$\mathrm{Ric}(\omega_0) \,-\, \omega_0 \,=\, \sqrt{-1}\,\partial\bar\partial\, h_0, \qquad \int_M (e^{h_0}-1)\,\omega_0^n \,=\, 0.$$

Note that the first equation above is equivalent to

$$\mathrm{Ric}(\omega_0) \,=\, \mu\,\omega_0 \,+\, 2\pi(1-\beta)[D] \,+\, \sqrt{-1}\,\partial\bar\partial(h_0 - (1-\beta)\,\log ||S||_0^2),$$

where S is a holomorphic section of $K_M^{-\lambda}$ defining D and $||\cdot||_0$ is a Hermitian norm on $K_M^{-\lambda}$ with $\lambda\,\omega_0$ as its curvature. For convenience, we assume that

$$\sup_M ||S||_0 = 1.$$

If ω_β and h_β are those in (1.3), then modulo a constant,

$$h_\beta \,=\, h_0 \,-\, (1-\beta)\log ||S||_0^2 \,-\, \log\left(\frac{\omega_\beta^n}{\omega_0^n}\right) \,-\, \mu\,\psi_\beta,$$

where $\omega_\beta = \omega_0 + \sqrt{-1}\,\partial\bar\partial\psi_\beta$.

It follows from (1.2)

$$(\omega_0 + \sqrt{-1}\,\partial\bar\partial\,\varphi)^n = e^{h_0-(1-\beta)\log||S||_0^2+a_\beta-\mu\varphi}\,\omega_0^n, \tag{2.1}$$

where a_β is chosen according to

$$\int_M \left(e^{h_0-(1-\beta)\log||S||_0^2+a_\beta} - 1\right)\omega_0^n = 0.$$

Clearly, a_β is uniformly bounded so long as $\beta \geq \beta_0 > 0$.

The Lagrangian $\mathbf{F}_{\omega_0,\mu}(\varphi)$ of (2.1) is given by

$$\mathbf{J}_{\omega_0}(\varphi) - \frac{1}{V}\int_M \varphi\,\omega_0^n - \frac{1}{\mu}\log\left(\frac{1}{V}\int_M e^{h_0-(1-\beta)\log||S||_0^2+a_\beta-\mu\varphi}\,\omega_0^n\right), \tag{2.2}$$

where $V = \int_M \omega_0^n$ and

$$\mathbf{J}_{\omega_0}(\varphi) = \frac{1}{V}\sum_{i=0}^{n-1}\frac{i+1}{n+1}\int_M \sqrt{-1}\,\partial\varphi\wedge\bar\partial\varphi\wedge\omega_0^i\wedge\omega_\varphi^{n-i-1}, \tag{2.3}$$

where $\omega_\varphi = \omega_0 + \sqrt{-1}\,\partial\bar\partial\,\varphi$. Note that $\mathbf{F}_{\omega_0,\mu}$ is well-defined for any continuous function φ.

Let us recall the following result

Theorem 2.1. *If $\omega = \omega_\varphi$ is a conic Kähler-Einstein with cone angle $2\pi\beta$ along D, then φ attains the minimum of the functional $\mathbf{F}_{\omega_0,\mu}$ on the space $\mathcal{K}_\beta(M,\omega_0)$ which consists of all smooth functions ψ on $M\backslash D$ such that ω_ψ is a conic Kähler metric with angle $2\pi\beta$ along D. In particular, $\mathbf{F}_{\omega_0,\mu}$ is bounded from below.*

One can find its proof in [Bo11] (also see [LS14]). An alternative proof may be given by extending the arguments in [DT91] to conic Kähler metrics.

Corollary 2.2. *If $\mu < 1$, then there are $\epsilon > 0$ and $C_\epsilon > 0$, which may depend on ω and μ, such that for any $\psi \in \mathcal{K}_\beta(M,\omega_0)$, we have for any $t \in (0,\mu]$*[5]

$$\mathbf{F}_{\omega_0,t}(\psi) \geq \epsilon\,\mathbf{J}_{\omega_0}(\psi) - C_\epsilon. \tag{2.4}$$

Proof. It follows from the arguments of using the log-α-invariant in [LS14] that $\mathbf{F}_{\omega_0,t}$ satisfies (2.4) for $t > 0$ sufficiently small. Let $\omega = \omega_\varphi$ be the conic Kähler-Einstein metric with angle $2\pi\beta$ along D. Then φ satisfies (2.1). Since M does not admit non-zero holomorphic fields[6], it follows from [Do10] that (2.1) has a solution $\bar\varphi$ when μ is replaced by $\bar\mu = \mu + \delta$ for $\delta >$ sufficiently small. Hence, by Theorem 2.1, $\mathbf{F}_{\omega_0,\bar\mu}$ is bounded from below. Then this corollary follows from Proposition 1.7 in [LS14] [7]

□

[5] The corresponding β_t is defined by $(1-t) = (1-\beta_t)\lambda$.

[6] Even if M does have non-trivial holomorphic fields, one can choose D such that there are no holomorphic fields which are tangent to D. This is sufficient for rest of the proof.

[7] In [LS14], the reference metric ω_0 may be slightly different from ours, however, the arguments apply.

Now we consider the following equation:

$$(\omega_0 + \sqrt{-1}\,\partial\bar\partial\,\varphi)^n \;=\; e^{h_\delta - \mu\varphi}\,\omega_0^n, \tag{2.5}$$

where

$$h_\delta \;=\; h_0 - (1-\beta)\log(\delta + ||S||_0^2) + c_\delta$$

for some constant c_δ determined by

$$\int_M \left(e^{h_0 - (1-\beta)\log(\delta+||S||_0^2)+c_\delta} - 1\right)\,\omega_0^n \;=\; 0.$$

Clearly, c_δ is uniformly bounded. If φ_δ is a solution, then we get a smooth Kähler metric

$$\omega_\delta \;=\; \omega_0 \;+\; \sqrt{-1}\,\partial\bar\partial\,\varphi_\delta.$$

Its Ricci curvature is given by

$$\mathrm{Ric}(\omega_\delta) \;=\; \mu\,\omega_\delta \;+\; \frac{\delta(1-\beta)\lambda}{\delta + ||S||_0^2}\,\omega_0 \;+\; \delta(1-\beta)\,\frac{\nabla S \wedge \overline{\nabla S}}{(\delta + ||S||_0^2)^2},$$

where ∇S denotes the covariant derivative of S with respect to the Hermitian metric $||\cdot||_0$. In particular, the Ricci curvature of ω_δ is greater than μ whenever $\beta < 1$ and $\delta > 0$.[8]

We will solve (2.5) for such ω_δ's and show that they converge to the conic Kähler-Einstein metric ω in a suitable sense.

To solve (2.5), we use the standard continuity method:

$$(\omega_0 + \sqrt{-1}\,\partial\bar\partial\,\varphi)^n \;=\; e^{h_\delta - t\varphi}\,\omega_0^n. \tag{2.6}$$

Define I_δ to be the set of $t \in [0,\mu]$ for which (2.6) is solvable. By the Calabi-Yau theorem, $0 \in I_\delta$.

We may assume $\mu < 1$, otherwise, we have nothing more to do.

Lemma 2.3. *The interval I_δ is open.*

Proof. If $t \in I_\delta$ and φ is a corresponding solution of (2.6), then the Ricci curvature of the associated metric ω_φ is equal to

$$t\,\omega_\varphi \;+\; \left((\mu - t) + \frac{\delta(1-\beta)\lambda}{\delta + ||S||_0^2}\right)\,\omega_0 \;+\; \delta(1-\beta)\,\frac{\nabla S \wedge \overline{\nabla S}}{(\delta + ||S||_0^2)^2}.$$

So $\mathrm{Ric}(\omega_\varphi) \;>\; t\,\omega_\varphi$. By the well-known Bochner identity, the first non-zero eigenvalue of ω_φ is strictly bigger than t. It implies that the linearization $\Delta_t + t$ of (2.6) at φ is invertible, where Δ_t is the Laplacian of ω_φ. By the Implicit Function Theorem, (2.6) is solvable for any t' close to t, so I_δ is open. □

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[8] This observation is crucial in our approximating the conic Kähler-Einstein metric ω. This and (2.5) first appeared in my lecture at SBU on October 25, 2012. Once we have this observation on Ricci curvature, the arguments in establishing the existence of ω_δ as in Theorem 2.5 and 2.6 below are identical to what I used in [Ti97].

Therefore, we only need to prove that I_δ is closed. This is amount to a priori estimates for any derivatives of the solutions of (2.6). As usual, by using known techniques in deriving higher order estimates, we need to bound only $J_{\omega_0}(\varphi)$ for any solution φ of (2.6) (cf. [Ti97], [Ti00]). The following arguments are identical to those for proving that the properness of $\mathbf{F}_{\omega_0,1}$ implies the existence of the Kähler-Einstein metrics in Theorem 1.6 of [Ti97].

We introduce

$$\mathbf{F}_{\delta,t}(\varphi) \,=\, \mathbf{J}_{\omega_0}(\varphi) \,-\, \frac{1}{V}\int_M \varphi\, \omega_0^n \,-\, \frac{1}{t}\log\left(\frac{1}{V}\int_M e^{h_\delta - t\varphi}\,\omega_0^n\right). \tag{2.7}$$

This is the Lagrangian of (2.6).

Lemma 2.4. *There is a constant C independent of t satisfying: For any smooth family of φ_s ($s \in [0,t]$) such that $\varphi = \varphi_t$ and φ_s solves* (2.6) *with parameter s, we have*

$$\mathbf{F}_{\delta,t}(\varphi) \,\leq\, C.$$

Proof. First we observe

$$\mathbf{F}_{\delta,s}(\varphi_s) \,=\, \mathbf{J}_{\omega_0}(\varphi_s) \,-\, \frac{1}{V}\int_M \varphi_s\,\omega_0^n. \tag{2.8}$$

So its derivative on s is given by

$$\frac{d}{ds}\,\mathbf{F}_{\delta,s}(\varphi_s) \,=\, \frac{1}{sV}\int_M \varphi_s\,(\omega_0 + \sqrt{-1}\,\partial\bar\partial\,\varphi_s)^n.$$

Here we have used the fact

$$\int_M (s\,\dot\varphi_s \,+\, \varphi_s)\,(\omega_0 + \sqrt{-1}\,\partial\bar\partial\,\varphi_s)^n \,=\, 0$$

This follows from differentiating (2.6) on s.

We will show that the derivative in (2.8) is bounded from above. Without loss of the generality, we may assume that $s \geq s_0 > 0$. Then we have

$$\mathrm{Ric}(\omega_{\varphi_s}) \,\geq\, s\,\omega_{\varphi_s} \,\geq\, s_0\,\omega_{\varphi_s},$$

and consequently, the Sobolev constant of ω_{φ_s} is uniformly bounded. By the standard Moser iteration, we have (cf. [Ti00])

$$-\inf_M \varphi_s \,\leq\, -\frac{1}{V}\int_M \varphi_s\,(\omega_0 + \sqrt{-1}\partial\bar\partial\,\varphi_s)^n \,+\, C'.$$

Since $\inf_M \varphi_s \,\leq\, 0$, we get

$$\frac{d}{ds}\,\mathbf{F}_{\delta,s}(\varphi_s) \,\leq\, s_0^{-1}C'.$$

The lemma follows from integration along s. □

Next we observe for any $t \le \mu$

$$h_\delta \,=\, h_0 - (1-\beta)\log(\delta + ||S||_0^2) + c_\delta \,\le\, h_0 - (1-\beta_t)\log ||S||_0^2 + c_\delta.$$

Hence, by Corollary 2.2, we have

$$\mathbf{F}_{\delta,t}(\psi) \,\ge\, \epsilon\,\mathbf{J}_{\omega_0}(\psi) \,-\, C_\epsilon \,-\, \frac{c_\delta - a_\beta}{t}.$$

Since both c_δ and a_β are uniformly bounded, combined with Lemma 2.4, we conclude that $J_{\omega_0}(\varphi)$ is uniformly bounded for any solution φ of (2.6).[9] Thus we have proved

Theorem 2.5. *For any $\delta > 0$, (2.5) has a unique smooth solution φ_δ. Consequently, we have a Kähler metric $\omega_\delta = \omega_0 + \sqrt{-1}\,\partial\bar\partial\varphi_\delta$ with Ricci curvature greater than or equal to μ.*

Next we examine the limit of ω_δ or φ_δ as δ tends to 0. First we note that for the conic Kähler-Einstein metric ω with cone angle $2\pi\beta$ along D given above, there is a uniform constant $c = c(\omega)$ such that $\sup_M |\varphi_\delta| \,\le\, c$. Using $\mathrm{Ric}(\omega_\delta) \,\ge\, \mu\,\omega_\delta$ and the standard computations, we have

$$\Delta \log \mathrm{tr}_{\omega_\delta}(\omega_0) \,\ge\, -a\,\mathrm{tr}_{\omega_\delta}(\omega_0),$$

where Δ is the Laplacian of ω_δ and a is a positive upper bound of the bisectional curvature of ω_0. If we put

$$u \,=\, \log \mathrm{tr}_{\omega_\delta}(\omega_0) \,-\, (a+1)\,\varphi_\delta,$$

then it follows from the above

$$\Delta u \,\ge\, e^{u-(a+1)\,c} \,-\, n(a+1).$$

Hence, we have

$$u \,\le\, (n+c)\,(a+1),$$

this implies

$$C^{-1}\,\omega_0 \,\le\, \omega_\delta,$$

where $C \,=\, (n+2\,c)(a+1)$. Using the equation (2.6), we have

$$C^{-1}\,\omega_0 \,\le\, \omega_\delta \,\le\, C'\,(\delta + ||S||^2)^{-(1-\beta)}\,\omega_0, \tag{2.9}$$

where C' is a constant depending only on a and ω_0. Since $\beta > 0$, the above estimate on $\omega_\delta = \omega_0 + \sqrt{-1}\,\partial\bar\partial\,\varphi_\delta$ gives the uniform Hölder continuity of φ_δ. Furthermore, using the Calabi estimate for the 3rd derivatives and the standard regularity theory, we can prove (cf. [Ti00]): For any $l > 2$ and a compact subset $K \subset M\backslash D$, there is a uniform constant $C_{l,K}$ such that

$$||\varphi_\delta||_{C^l(K)} \,\le\, C_{l,K}. \tag{2.10}$$

Then we can deduce from the above estimates:

[9] Here we also used the fact that $J_{\omega_0}(\varphi)$ is automatically bounded for $t > 0$ sufficiently small.

Theorem 2.6. *The smooth Kähler metrics ω_δ converge to ω in the Gromov-Hausdorff topology on M and in the smooth topology outside D.*

Proof. It suffices to prove the first statement: ω_δ converge to ω in the Gromov-Hausdorff topology. Since ω_δ has Ricci curvature bounded from below by a fixed $\mu > 0$, by the Gromov Compactness Theorem, any sequence of (M, ω_δ) has a subsequence converging to a length space $(\bar{M}, \bar{d})$ in the Gromov-Hausdorff topology. We only need to prove that any such a limit $(\bar{M}, \bar{d})$ coincides with (M, ω). Without loss of generality, we may assume that (M, ω_δ) converge to $(\bar{M}, \bar{d})$ in the Gromov-Hausdorff topology. By the estimates on derivatives in (2.10), $\bar{M}$ contains an open subset U which can be identified with $M \backslash D$, moreover, this identification $\iota : M\backslash D \mapsto U$ is an isometry between $(M\backslash D, \omega|_{M\backslash D})$ and $(U, \bar{d}|_U)$. On the other hand, since ω is a conic metric with angle $2\pi\beta \leq 2\pi$ along D, one can easily show by standard arguments that M M is the metric completion of $M\backslash D$ with respect to ω. Then it follows from (2.9) that ι extends to a Lipschtz map from (M, ω) onto $(\bar{M}, \bar{d})$, still denoted by ι. In fact, the Lipschtz constant is 1.

We claim that ι is an isometry. This is equivalent to the following: For any p and q in $M\backslash D$,

$$d_\omega(p, q) \;=\; \bar{d}(\iota(p), \iota(q)).$$

It also follows from (2.9) that $\bar{D} = \iota(D)$ has Hausdorff measure 0 and is the Gromov-Hausdorff limit of D under the convergence of (M, ω_δ) to $(\bar{M}, \bar{d})$. To prove the above claim, we only need to prove: For any $\bar{p}, \bar{q} \in \bar{M}\backslash \bar{D}$, there is a minimizing geodesic $\gamma \subset \bar{M}\backslash\bar{D}$ joining $\bar{p}$ to $\bar{q}$. Its proof is based on a relative volume comparison estimate due to Gromov ([Gr97], p 523, (B)). [10] We will prove it by contradiction. If no such a geodesic joins $\bar{p}$ to $\bar{q}$, then, observing that $M\backslash D$ is geodesically convex with respect to ω, we have

$$\bar{d}(\bar{p}, \bar{q}) \;<\; d_\omega(p, q),$$

where $\bar{p} = \iota(p)$ and $\bar{q} = \iota(q)$. Then there is a $r > 0$ satisfying:

(1) $B_r(\bar{p}, \bar{d}) \cap \bar{D} = \emptyset$ and $B_r(\bar{q}, \bar{d}) \cap \bar{D} = \emptyset$, where $B_r(\cdot, \bar{d})$ denotes a geodesic ball in $(\bar{M}, \bar{d})$;

(2) $\bar{d}(\bar{x}, \bar{y}) < d_\omega(x, y)$, where $\bar{x} = \iota(x) \in B_r(\bar{p}, \bar{d})$ and $\bar{y} = \iota(y) \in B_r(\bar{q}, \bar{d})$.

It follows from (1) and (2) that any minimizing geodesic γ from $\bar{x}$ to $\bar{y}$ intersects with $\bar{D}$. By choosing r sufficiently small, we may have

$$B_r(\bar{p}, \bar{d}) = \iota(B_r(p, \omega)) \quad \text{and} \quad B_r(\bar{q}, \bar{d}) = \iota(B_r(q, \omega)).$$

Choose a small tubular neighborhood T of D in M whose closure is disjoint from both $B_r(p, \omega)$ and $B_r(q, \omega)$. It is easy to see that T can be chosen to have the

[10] I am indebted to Jian Song for this reference. He seems to be the first of applying such an estimate to studying the convergence problem in Kähler geometry.

volume of ∂T as small as we want. Now we choose $p_\delta, q_\delta \in M$ and neighborhood T_δ of D with respect to ω_δ such that in the Gromov-Haudorff convergence,

$$\lim_{\delta\to 0+} p_\delta \,=\, \bar p\,, \quad \lim_{\delta\to 0+} q_\delta \,=\, \bar q\,, \quad \lim_{\delta\to 0+} T_\delta \,=\, \iota(T)\,.$$

It follows

$$\lim_{\delta\to 0+} Vol(\partial T_\delta, \omega_\delta) \,=\, Vol(\partial T, \omega).$$

Also, for δ sufficiently small, $B_r(p_\delta, \omega_\delta)$, $B_r(q_\delta, \omega_\delta)$ and T_δ are mutually disjoint. Clearly, any minimizing geodesic γ_δ from any $w \in B_r(p_\delta, \omega_\delta)$ to $z \in B_r(q_\delta, \omega_\delta)$ intersects with T_δ, so by Gromov's estimate ([Gr97], p523, (B)),

$$c\, r^{2n} \,\le\, Vol(B_r(q_\delta, \omega_\delta), \omega_\delta) \,\le\, C\, Vol(\partial T_\delta, \omega_\delta),$$

where c depends only on β and C depends only on β, n, r. This leads to a contradiction because $Vol(\partial T_\delta, \omega_\delta)$ converge to $Vol(\partial T, \omega)$ which can be made as small as we want. Thus, ι is an isometry and our theorem is proved.

□

3 An extension of Cheeger-Colding-Tian

In this section, we show a compactness theorem on conic Kähler-Einstein metrics. This theorem, coupled with the smooth convergence result in the next section, extends a result of Cheeger-Colding-Tian [CCT02] on smooth Kähler-Einstein metrics. In fact, our proof makes use of results in [CCT02] with injection of some new technical ingredients.

Let ω_i be a sequence of conic Kähler-Einstein metrics with cone angle $2\pi\beta_i$ along D, so we have

$$\mathrm{Ric}(\omega_i) \,=\, \mu_i\omega_i \,+\, 2\pi(1-\beta_i)\,[D], \qquad \mu_i \,=\, 1-(1-\beta_i)\lambda.$$

We assume that $\lim \beta_i = \beta_\infty > 1-\lambda^{-1}$, it follows $\lim \mu_i = \mu_\infty > 0$.

For each ω_i, we use Theorem 2.6 to get a smooth Kähler metric $\tilde\omega_i$ satisfying:

A1. Its Kähler class $[\tilde\omega_i] \,=\, 2\pi c_1(M)$;

A2. Its Ricci curvature $\mathrm{Ric}(\tilde\omega_i) \,\ge\, \mu_i\, \tilde\omega_i$;

A3. The Gromov-Hausdorff distance $d_{GH}(\omega_i, \tilde\omega_i)$ is less that $1/i$.

By the Gromov compactness theorem, a subsequence of $(M, \tilde\omega_i)$ converges to a metric space (M_∞, d_∞) in the Gromov-Hausdorff topology. For simplicity, we may assume that $(M, \tilde\omega_i)$ converges to (M_∞, d_∞). It follows from **A3** above that (M, ω_i) also converges to (M_∞, d_∞) in the Gromov-Hausdorff topology.

Theorem 3.1. *There is a closed subset $\mathcal{S} \subset M_\infty$ of Hausdorff codimension at least 2 such that $M_\infty\backslash\mathcal{S}$ is a smooth Kähler manifold and d_∞ is induced by a Kähler-Einstein metric ω_∞ outside $\mathcal{S}$, that is,*

$$\mathrm{Ric}(\omega_\infty) \,=\, \mu_\infty\, \omega_\infty \quad \text{ on } \; M_\infty\backslash\mathcal{S}.$$

If $\beta_\infty < 1$, then ω_i converges to ω_∞ in the C^∞-topology outside $\mathcal{S}$. Moreover, if $\beta_\infty = 1$, the set $\mathcal{S}$ is of codimension at least 4 and ω_∞ extends to a smooth Kähler metric on $M_\infty \backslash \mathcal{S}$.

This theorem is essentially due to Z.L. Zhang and myself [TZ12]. In this joint work, we develop a regularity theory for conic Einstein metrics which generalizes the work of Cheeger-Colding and Cheeger-Colding-Tian. Here, for completion and convenience, we give an alternative proof by using the approximations from last section.

Proof. Using the fact that (M_∞, d_∞) is the Gromov-Hausdorff limit of $(M, \tilde\omega_i)$, we can deduce from [CC95] the existence of tangent cones at every $x \in M_\infty$. More precisely, given any $x \in M_\infty$, for any $r_i \mapsto 0$, by taking a subsequence if necessary, $(M_\infty, r_i^{-2} d_\infty, x)$ converges to a tangent cone $\mathcal{C}_x$ at x. Define $\mathcal{R}$ to be the set of all points x in M_∞ such that some tangent cone $\mathcal{C}_x$ is isometric to $\mathbb{R}^{2n}$.

First we prove that $\mathcal{R}$ is open. If $\beta_\infty = 1$, then $\lim \mu_i = 1$. Since

$$[\tilde\omega_i] \,=\, 2\pi\, c_1(M) \quad \text{and} \quad \mathrm{Ric}(\tilde\omega_i) \,\ge\, \mu_i\, \tilde\omega_i,$$

we have (cf. Appendix 2)

$$\int_M |\mathrm{Ric}(\tilde\omega_i) \,-\, \tilde\omega_i|\, \tilde\omega_i^n \,\le\, 2\,(1-\mu_i) \int_M \tilde\omega_i^n \,\to\, 0. \tag{3.1}$$

This means that $(M, \tilde\omega_i)$ form a sequence of almost Kähler-Einstein metrics in the sense of [TW12].[11] Then it follows from Theorem 2 in [TW12] (also Theorem 8.1 in Appendix 2) that M_∞ is smooth outside a closed subset $\mathcal{S}$ of codimension at least 4 and d_∞ is induced by a smooth Kähler-Einstein metric ω_∞ on $M_\infty \backslash \mathcal{S}$.

Now assume that $\beta_\infty < 1$. Note that (M, ω_i) also converge to (M_∞, d_∞) in the Gromov-Hausdorff topology. Let $\{x_i\}$ be a sequence of points in M which converge to $x \in \mathcal{R}$ during (M, ω_i)'s converging to (M_∞, d_∞). Since $x \in \mathcal{R}$, there is a tangent cone $\mathcal{C}_x$ of (M_∞, d_∞) at x which is isometric to $\mathbb{R}^{2n}$. It follows that for any $\epsilon > 0$, there is a $r = r(\epsilon)$ such that

$$\frac{Vol(B_r(x, d_\infty))}{r^{2n}} \,\ge\, c(n) - \epsilon,$$

where $c(n)$ denotes the volume of the unit ball in $\mathbb{R}^{2n}$. On the other hand, if $y_i \in D$, then by the Bishop-Gromov volume comparison, for any $\tilde r > 0$, we have

$$\frac{Vol(B_{\tilde r}(y_i, \omega_i))}{\tilde r^{2n}} \,\le\, c(n)\, \beta_i.$$

It also follows from the Bishop-Gromov volume comparison that there is an $N = N(\epsilon)$ such that for any small $\bar r \in (0, r/N)$ and $y_i \in B_{\bar r}(x_i, \omega_i)$, we have

$$1 - \epsilon \,\le\, \frac{Vol(B_r(y_i, \omega_i))}{Vol(B_r(x_i, \omega_i))} \,\le\, 1 + \epsilon.$$

[11] To see why we call $(M, \tilde\omega_i)$ a sequence of almost Kähler-Einstein metrics, we first note that if $(M, \tilde\omega_i)$ has a smooth limit, then such a limit must be a Kähler-Einstein metric. However, (3.1) indicates that $(M, \tilde\omega_i)$ converges to a Kähler-Einstein metric in the L^1-sense.

Now we claim that if $\bar{r} = r/N$, we have $B_{\bar{r}}(x_i, \omega_i) \cap D = \emptyset$. If this claim is false, say $y_i \in B_{\bar{r}}(x_i, \omega_i) \cap D$, then for i sufficiently large, we can deduce from the above and a result of Colding [Co94] on the volume convergence in the Gromov-Hausdorff topology

$$c(n) - 2\epsilon \leq \frac{Vol(B_r(x_i, \omega_i))}{r^{2n}} \leq (1+\epsilon) \frac{Vol(B_r(y_i, \omega_i))}{r^{2n}} \leq c(n)(1+\epsilon)\,\beta_i.$$

Then we get a contradiction if ϵ is chosen sufficiently small. The claim is proved.

Since $B_{\bar{r}}(x_i, \omega_i)$ is contained in the smooth part of (M, ω_i) and its volume is sufficiently close to that of an Euclidean ball, the curvature of ω_i is uniformly bounded on the smaller ball $B_{3\bar{r}/4}(x_i, \omega_i)$ (cf. [An90]). It follows that ω_i restricted to $B_{\bar{r}/2}(x_i, \omega_i)$ converge to a smooth Kähler-Einstein metric on $B_{\bar{r}/2}(x, d_\infty)$ and $B_{\bar{r}/2}(x, d_\infty) \subset \mathcal{R}$. So $\mathcal{R}$ is open and d_∞ restricted to $\mathcal{R}$ is induced by a smooth Kähler-Einstein metric ω_∞.

The rest of the proof is standard in view of [CCT02].

Let $\mathcal{S}_k$ ($k = 0, 1, \cdots, 2n-1$) denote the subset of M_∞ consisting of points for which no tangent cone splits off a factor, $\mathbb{R}^{k+1}$, isometrically. Clearly, $\mathcal{S}_0 \subset \mathcal{S}_1 \subset \cdots \subset \mathcal{S}_{2n-1}$. It is proved by Cheeger-Colding [CC95] that $\mathcal{S}_{2n-1} = \emptyset$, $\dim \mathcal{S}_k \leq k$ and $\mathcal{S} = \mathcal{S}_{2n-2}$. Moreover, if $\beta_\infty = 1$, it follows from [TW12] or Appendix 2 that $\mathcal{S} = \mathcal{S}_{2n-4}$. Then we have proved this theorem. □

Using the same arguments in [CCT02], one can show:

Theorem 3.2. *Let $\mathcal{C}_x$ be a tangent cone of M_∞ at $x \in \mathcal{S}$, then we have*

C1. *Each $\mathcal{C}_x$ is regular outside a closed subcone $\mathcal{S}_x$ of complex codimension at least 1. Such a $\mathcal{S}_x$ is the singular set of $\mathcal{C}_x$;*

C2. *$\mathcal{C}_x = \mathbb{C}^k \times \mathcal{C}'_x$, in particular, $\mathcal{S}_{2k+1} = \mathcal{S}_{2k}$. We will denote by o the vertex of $\mathcal{C}_x$;*

C3. *There is a natural Kähler Ricci-flat metric g_x whose Kähler form ω_x is $\sqrt{-1}\,\partial\bar{\partial}\rho_x^2$ on $\mathcal{C}_x \backslash \mathcal{S}_x$, where ρ_x denotes the distance function from o. Also g_x is a cone metric;*

C4. *For any $x \in \mathcal{S}_{2n-2}$ with $\mathcal{C}_x = \mathbb{C}^{n-1} \times \mathcal{C}'_x$, then $\mathcal{C}'_x$ is a 2-dimensional flat cone of angle $2\pi\bar{\beta}$ such that $0 < \bar{\beta}_\infty \leq \bar{\beta} \leq \beta_\infty$ and $(1-\bar{\beta}) = m(1-\beta_\infty)$ for some integer $m \geq 1$, where $\bar{\beta}_\infty$ depends only on β_∞ and $c_1(M)^n$.*

Proof. **C1**, **C2** and **C3** follow directly from results in [CCT02]. [12] The proof of **C4** uses the slicing argument in [CCT02] (also see [Ch03]) for proving that $\mathcal{S}_{2n-2} = \emptyset$ in the case of smooth Kähler-Einstein metrics. For the readers' convenience, we adapt the arguments for a proof in our case.

Since (M, ω_i) converge to $(M_\infty, \omega_\infty)$, there are $r_i \to 0$ and $x_i \in M$ such that $(M, r_i^{-1}\omega_i, x_i)$ converge to the cone $\mathcal{C}_x$. It follows from Theorem 2.37 in

[12] To prove that g_x is Kähler-Ricci flat in **C3**, we use the fact that (M_∞, d_∞) is also the limit of conic Kähler-Einstein manifolds (M, ω_i).

[CCT02] that there are $\epsilon_i \to 0$ and maps $(\Phi_i, \mathbf{u}_i) : B_{5/2}(x_i, r_i^{-2}\omega_i) \mapsto \mathbb{C}^{n-1} \times \mathbb{R}_+$ satisfying:

$$\begin{aligned} \max\{\mathrm{Lip}(\Phi_i),\, \mathrm{Lip}(\mathbf{u}_i)\} &\le c(n), \\ \int_{|z|<1,\, z\in\mathbb{C}^{n-1}} \big|V(z) \,-\, 2\pi\bar\beta\big| \, dz \wedge d\bar z &\le \epsilon_i, \end{aligned}$$

where $V(z)$ is the volume of $\Sigma_z = \Phi_i^{-1}(z) \cap \mathbf{u}_i^{-1}([0,1])$ with respect to $r_i^{-2}\omega_i$. These correspond to (2.38) and (2.40) in [CCT02]. Actually, we first apply Theorem 2.37 in [CCT02] to smooth approximations of (M, ω_i) produced in Theorem 2.6 and then take the limit. Moreover, in view of the proof of Theorem 2.37 in [CCT02], the components of Φ_i and $\mathbf{u}_i$ are defined by solving Laplacian equations, so they are smooth outside D. By slight modification if necessary, we may also assume that Φ_i is smooth along D.

In the following, for simplicity, write $\epsilon = \epsilon_i$ and $(\Phi, \mathbf{u}) = (\Phi_i, \mathbf{u}_i)$. As a consequence of the above estimates on Φ and $\mathbf{u}$, we can find a subset $\mathbf{B}_\epsilon$ of $\{\, |z| < 1 \,\} \subset \mathbb{C}^{n-1}$ with large measure such that for any $z \in \mathbf{B}_\epsilon$, Σ_z is transversal to D with its boundary converging to $\{z\} \times S^1_{\bar\beta}$ as $i \to \infty$, where $S^1_{\bar\beta}$ denotes the unit circle in $\mathcal{C}'_x$, and

$$|V(z) \,-\, 2\pi\bar\beta| \,\le\, C\,\epsilon,$$

where C is a uniform constant. Now K_M^{-1} restricts to a line bundle on Σ_z with am induced Hermitian metric h_z by $r_i^{-2}\omega_i$ whose curvature Ω is equal to

$$\mathrm{Ric}(r^{-2}\omega_i) \,=\, \mu\,\omega_i \,+\, 2\pi(1-\beta_i)\,\iota_z^*[D],$$

where $\iota_z : \Sigma_z \mapsto M$ denotes the embedding. Let $\pi : S\Sigma_z \mapsto \Sigma_z$ be the unit circle bundle of this Hermitian line bundle, then

$$\pi^*\Omega \,=\, d\,\theta \quad \text{ on } \ \Sigma_z \backslash D,$$

where θ denotes the connection 1-form of h_z which has residue equal to $\pm\,(1-\beta_i)$ at each intersection of Σ_z with D.[13] Since z is a regular value, the normal bundle of Σ_z is trivial, so the Euler number of K_M^{-1} restricted to Σ_z is the same as that of $T\Sigma_z$. It follows that there is a section v of K_M^{-1} over Σ_z with non-degenerate zeroes outside $D \cap \Sigma_z$ and which is equal to outward unit normal of $\partial\Sigma_z$ along the boundary of Σ_z. Note that

$$\chi(\Sigma_z) \,=\, \sum_{v(p)=0} \pm 1 \,\le\, 1.$$

Put

$$s = \frac{v}{||v||} : \Sigma_z \backslash (D \cup v^{-1}(0)) \mapsto S\Sigma_z.$$

Hence, by the Stokes Theorem, we have

$$\int_{\Sigma_z} \Omega \,=\, \int_{\partial\Sigma_z} s^*\theta \,-\, \sum_{p\,\in\, D \ \text{ or } \ v(p)=0} \lim_{\delta\to 0} \int_{\partial B_\delta(p, r_i^{-2}\omega_i)} s^*\theta$$

[13]The sign depends on whether or not Σ_z intersects with D positively.

It follows

$$(\chi(\Sigma_z) - \bar\beta) \, - \, m \, (1 - \beta_\infty) \, = \, o(1),$$

where $m = m(z)$ is the algebraic intersection number of Σ_z with D and $o(1)$ denotes a quantity which converges to 0 as i goes to ∞. If m is non-negative, then we see $1 - \bar\beta = m(1 - \beta_\infty)$ by taking i sufficiently large.

Now we claim that there are regular values $z's$ of Φ such that $m(z) \geq 0$. This follows from the co-area formula:[14]

$$\int_{B_6(x_i, r_i^{-2}\omega_i) \cap D} |d\Phi| \, \omega_i^{n-1} \, = \, \int_{z \in \mathbb{C}^{n-1}} m(z) \, dz \wedge d\bar z.$$

The left side is non-negative, so we can find a subset of regular values in $\mathbb{C}^{n-1}$ of positive measure such that $m(z) \geq 0$. Our claim is proved.

The bounds on $\bar\beta$ follow from the Bishop-Gromov volume comparison. Note that $\bar\beta_\infty$ depends only on the diameter and volume of M_∞. Hence, there are only finitely many of such $\bar\beta$ if $\beta_\infty < \infty$. □

Next we state another corollary of Theorem 2.6:

Lemma 3.3. *There is a uniform bound on the Sobolev constants of (M, ω_i), that is, there is a constant C such that for any $f \in C^1(M, \mathbb{R})$,*

$$\left(\int_M |f|^{\frac{2n}{n-1}} \, \omega_i^n \right)^{\frac{n-1}{n}} \, \leq \, C \int_M (|df|^2_{\omega_i} \, + \, |f|^2) \, \omega_i^n. \tag{3.2}$$

Proof. By Theorem 2.6, for any i, there is a sequence of smooth Kähler metrics $\omega_{i,\delta}$ converging to ω_i in the Gromov-Hausdorff topology and $\mathrm{Ric}(\omega_{i,\delta}) \, \geq \, \mu_i \, \omega_{i,\delta}$. Since the volume of $\omega_{i,\delta}$ is fixed, it is well-known that (3.2) holds uniformly for $\omega_{i,\delta}$. Then the lemma follows by taking $\delta \to 0$ and applying results in [Ch99] and [HK95] to our special case. Actually, we can give a direct proof in our case. Let us indicate how to do it. First, by (2.9), ω_i and $\omega_{i,\delta}$ are all bounded from below by a smooth metric on M, so $|\nabla f|_{\omega_i}$ and $|\nabla f|_{\omega_{i,\delta}}$ are uniformly bounded from above by a constant which may depend on f. Secondly, as δ goes to 0, $\omega_{i,\delta}$ converge to ω_i in the smooth topology outside D. Then (3.2) follows easily. □

4 Smooth convergence

We will adopt the notations from last section, e.g., ω_i is a conic Kähler-Einstein metric on M with angle $2\pi\beta_i$ along D as before. The main result of this section is to show that ω_i converge to ω_∞ outside a close subset of codimension at least 2. This is crucial for our establishing the partial C^0-estimate for conic Kähler-Einstein metrics as well as finishing the proof Theorem 1.1. This is related to the limit of D when (M, ω_i) converges to (M_∞, d_∞). If $\beta_\infty < 1$, the limit of D is in the singular set $\mathcal{S}$ of M_∞ since ω_i converge to ω_∞ in the

[14] In Appendix 1, in our case of conic Kähler-Einstein metrics, we give an alternative way of constructing a slice Σ_z whose $m(z)$ is automatically positive.

C^∞-topology outside $\mathcal{S}$ as shown in Theorem 3.1. The difficulty lies in the case when $\beta_\infty = 1$. By [TW12] or Theorem 8.1 in Appendix 2, $\mathcal{S}$ is a closed subset of codimension at least 4, equivalently, M_∞ is actually smooth outside a closed subset of codimension 4. Related results for smooth Kähler-Einstein metrics were proved before (cf. [CCT02], [Ch03]). However, a priori, it is not even clear if ω_i converge to ω_∞ in a stronger topology on any open subset of $M_\infty \backslash \mathcal{S}$. The original arguments in [CCT02] rely on an argument in [An90] which works only for smooth metrics. It fails for conic Kähler-Einstein metrics. So we need to have a new approach. In the course of proving our main result in this section, we also exam the limit of D in M_∞.

First we describe a general and important construction: Given any conic metric ω with cone angle $2\pi\beta$ along D, its determinant gives a Hermitian metric $\tilde{H}$ on K_M^{-1} outside D. For simplicity, we will also denote by $\tilde{H}$ the induced Hermitian metric on $K_M^{-\ell}$ for any $\ell > 0$. However, $\tilde{H}$ is singular along D, more precisely, if S is a defining section of D, then it is of the order $||S||_0^{-2(1-\beta)}$ along D, where $||\cdot||_0$ is a fixed Hermitian norm. This implies that $\tilde{H}(S,S)^{\frac{1-\beta}{\mu}}\tilde{H}$ is bounded along D, where $\mu = 1-(1-\beta)\lambda$. On the other hand, there is a unique h such that as currents,

$$\mathrm{Ric}(\omega) \,=\, \mu\,\omega \,+\, 2\pi(1-\beta)\,[D] \,+\, \sqrt{-1}\,\partial\bar{\partial}h,$$

where h is normalized by

$$\int_M \left(e^h - 1\right)\,\omega^n \,=\, 0.$$

Note that h is Hölder continuous. Put

$$H_\omega(\cdot,\cdot) \,=\, e^{\frac{h}{\mu}}\,\tilde{H}(S,S)^{\frac{1-\beta}{\mu}}\,\tilde{H}(\cdot,\cdot),$$

then as a current, the curvature of H_ω is equal to

$$\mathrm{Ric}(\omega) \,-\, \frac{1-\beta}{\mu}\,\sqrt{-1}\,\partial\bar{\partial}\log\tilde{H}(S,S) \,-\, \frac{\sqrt{-1}}{\mu}\,\partial\bar{\partial}h \,=\, \omega.$$

Also we normalize H_ω by scaling S such that

$$\int_M H_\omega(S,S)\,\omega^n \,=\, \int_M e^{\frac{\lambda h}{\mu}}\,\tilde{H}(S,S)^{\frac{1}{\mu}}\,\omega^n \,=\, 1.$$

Such a Hermitian metric H_ω is uniquely determined by ω and D and called the associated Hermitian metric of ω. If ω is conic Kähler-Einstein, its associated metric H_ω is determined by the volume form ω^n, e.g., in local holomorphic coordinates $z_1,\cdots,z_n$, write

$$\omega \,=\, \sqrt{-1}\,\sum_{a,b=1}^n g_{a\bar{b}}\,dz_a\wedge d\bar{z}_b \quad\text{and}\quad S \,=\, f\,\left(\frac{\partial}{\partial z_1}\wedge\cdots\wedge\frac{\partial}{\partial z_n}\right)^\lambda,$$

then H_ω is represented by

$$\det(g_{a\bar{b}})^{\frac{1}{\mu}}\,|f|^{\frac{2(1-\beta)}{\mu}}.$$

In particular, it implies that for any $\sigma \in H^0(M, K_M^{-\ell})$, $H_\omega(\sigma,\sigma)$ is bounded along D.

Now we recall some identities for pluri-anti-canonical sections.

Lemma 4.1. *Let ω_i be as above and H_i be the associated Hermitian metric on K_M^{-1}. Then for any $\sigma \in H^0(M, K_M^{-\ell})$, we have (in the sense of distribution)*

$$\Delta_i ||\sigma||_i^2 \,=\, ||\nabla\sigma||_i^2 \,-\, n\ell\,||\sigma||_i^2 \tag{4.1}$$

and

$$\Delta_i ||\nabla\sigma||_i^2 \,=\, ||\nabla^2\sigma||_i^2 \,-\, ((n+2)\,\ell \,-\, \mu_i)\,||\nabla\sigma||_i^2, \tag{4.2}$$

where $||\cdot||_i$ denotes the Hermitian norm on $K_M^{-\ell}$ induced by $H_i = H_{\omega_i}$, ∇ denotes the covariant derivative of H_i and Δ_i denotes the Laplacian of ω_i.

Proof. On $M\backslash D$, both (4.1) and (4.2) were already derived in [Ti91] by direct computations. Since $||\sigma||_i^2$ is bounded, (4.1) holds on M.

By a direct computation in local coordinates, one can also show that $||\nabla\sigma||_i^2$ is bounded along D, so (4.2) also holds. □

Applying the standard Moser iteration to (4.1) and (4.2) and using Lemma 3.3, we obtain

Corollary 4.2. *There is a uniform constant C such that for any $\sigma \in H^0(M, K_M^{-\ell})$, we have*

$$\sup_M \left(||\sigma||_i \,+\, \ell^{-\frac{1}{2}}\,||\nabla\sigma||_i\right) \,\le\, C\,\ell^{\frac{n}{2}} \left(\int_M ||\sigma||_i^2\,\omega_i^n\right)^{\frac{1}{2}}. \tag{4.3}$$

If σ_i is a sequence in $H^0(M, K_M^{-\ell})$ satisfying:

$$\int_M ||\sigma_i||_i^2\,\omega_i^n \,=\, 1,$$

then by Corollary 4.2, $||\sigma_i||_i$ and their derivative are uniformly bounded. It implies that $||\sigma_i||_i$ are uniformly continuous. Hence, by taking a subsequence if necessary, we may assume $||\sigma_i||_i$ converge to a Lipschtz function F_∞ as i tends to ∞, moreover, we have

$$\int_{M_\infty} F_\infty^2\,\omega_\infty^n \,=\, 1.$$

In particular, F_∞ is non-zero. Our strategy is to prove that ω_i converge to ω_∞ on $F_\infty^{-1}(0)\cup\mathcal{S}$ and F_∞ is equal to the square norm of a holomorphic section on M_∞.

Now we assume $\sigma_i = a_i S$, where a_i are constants and S is a defining section of D. Then $||\sigma_i||_i(x) = 0$ if and only if $x \in D$. If $F_\infty(x) \ne 0$ for some $x \in M_\infty\backslash\mathcal{S}$, then for a sufficiently small $r > 0$, we have

$$2\,F_\infty(y) \,\ge\, F_\infty(x) \,>\, 0, \quad \forall\, y \in B_r(x, \omega_\infty).$$

This is because F_∞ is continuous. We can also have

$$B_r(x, \omega_\infty) \subset M_\infty \backslash \mathcal{S}.$$

Since $||\sigma_i||_i$ converge to F_∞ uniformly, for i sufficiently large, $||\sigma_i||_i > 0$ on those geodesic balls $B_r(x_i, \omega_i)$ of (M, ω_i) which converge to $B_r(x, \omega_\infty)$ in the Gromov-Hausdorff topology. It follows that $B_r(x_i, \omega_i) \subset M \backslash D$, that is, each $B_r(x_i, \omega_i)$ lies in the smooth part of (M, ω_i). On the other hand, since x is a smooth point of M_∞, by choosing smaller r, we can make the volume of $B_r(x_i, \omega_i)$ sufficiently close to that of corresponding Euclidean ball, then as one argued in [CCT02] by using a result of [An90], ω_i restricted to $B_r(x_i, \omega_i)$ converge to ω_∞ on any compact subset of $B_r(x, \omega_\infty)$ in the C^∞-topology. Thus, ω_i converge to ω_∞ in the C^∞-topology on the non-empty open subset $M_\infty \backslash F_\infty^{-1}(0) \cup \mathcal{S}$.

Next we want to show that $F_\infty^{-1}(0)$ does not contain any open subset,[15] or equivalently, $M_\infty \backslash F_\infty^{-1}(0)$ is an open-dense subset in M_∞. We prove it by contradiction. If it is false, say $U \subset F_\infty^{-1}(0)$ is open, using the fact that $||\sigma_i||_i$ is uniformly bounded from above, we have

$$\lim_{i \to \infty} \int_M \log(\frac{1}{i} + ||\sigma_i||_i^2)\, \omega_i^n \;=\; -\infty. \tag{4.4}$$

By a direct computation, we have

$$\omega_i \;+\; \sqrt{-1}\partial\bar\partial \log(\frac{1}{i} + ||\sigma_i||_i^2) \;=\; \frac{\omega_i}{1 + i\,||\sigma_i||_i^2} \;+\; \frac{i\,\nabla\sigma_i \wedge \overline{\nabla\sigma_i}}{(1 + i\,||\sigma_i||_i^2)^2} \;\geq\; 0.$$

It implies

$$\Delta_i \log(\frac{1}{i} + ||\sigma_i||_i^2) \;\geq\; -n.$$

Using the Sobolev inequality in Lemma 3.3 and the Moser iteration, we can deduce

$$\sup_M \log(\frac{1}{i} + ||\sigma_i||_i^2) \;\leq\; C\left(1 \;+\; \int_M \log(\frac{1}{i} + ||\sigma_i||_i^2)\, \omega_i^n\right), \tag{4.5}$$

where C is a uniform constant. By (4.4),

$$\lim_{i \to \infty} \sup_M \log(\frac{1}{i} + ||\sigma_i||_i^2) \;=\; -\infty.$$

However, since the L^2-norm of $||\sigma_i||_i$ is equal to 1, there is a constant c independent of i such that

$$\sup_M \log(\frac{1}{i} + ||\sigma_i||_i^2) \;\geq\; -c. \tag{4.6}$$

This leads to a contradiction. Therefore, $M_\infty \backslash F_\infty^{-1}(0)$ is dense.

[15] The same arguments actually show that it does not contain any subset of positive measure.

By our definition of the metric H_i associated to ω_i, in local holomorphic coordinates $z_1, \cdots, z_n$ away from D, we have

$$||\sigma_i||_i^2 \;=\; \Big((\det(g_{a\bar b}))^\lambda \, |w|^2 \Big)^{\frac{1}{\mu}}$$

where

$$\sigma_i \;=\; w \left(\frac{\partial}{\partial z_1} \wedge \cdots \wedge \frac{\partial}{\partial z_n} \right)^{\otimes \lambda} \quad \text{and} \quad \omega_i \;=\; \sqrt{-1} \sum_{a,b=1}^{n} g_{a\bar b}\, dz_a \wedge d\bar z_b \,.$$

Since ω_i converge to ω_∞ in the C^∞-topology outside $F_\infty^{-1}(0) \cup \mathcal{S}$, it follows that σ_i converge to a holomorphic section σ_∞ and $||\cdot||_i$ converge to a Hermitian norm $||\cdot||_\infty$ on $M \backslash F_\infty^{-1}(0) \cup \mathcal{S}$.[16] Note that $||\cdot||_\infty$ is the Hermitian norm on $K_{M_\infty}^{-1}$ associated to ω_∞. Clearly, $F_\infty = ||\sigma_\infty||_\infty$, in particular, σ_∞ is bounded. One can show that it extends to a holomorphic section of $K_{M_\infty}^{-\lambda}$ on the regular part $M_\infty \backslash \mathcal{S}$. For the reader's convenience, we show how to do such an extension. This extension is a local problem, so it suffices to extend σ_∞ near each $x \in M_\infty \backslash \mathcal{S}$. First we observe that (4.5) and (4.6) imply

$$\int_{M_\infty} \log F_\infty \, \omega_\infty^n \;\geq\; -C' \;>\; -\infty, \tag{4.7}$$

where C' is a uniform constant. Let $(U; z_1, \cdots, z_n)$ be a local holomorphic coordinates chart of M_∞ near x, then as above, on $U \backslash F_\infty^{-1}(0)$, we write

$$\sigma_\infty \;=\; w_\infty \left(\frac{\partial}{\partial z_1} \wedge \cdots \wedge \frac{\partial}{\partial z_n} \right)^{\otimes \lambda} .$$

Since $F_\infty = ||\sigma_\infty||_\infty$, by putting $w_\infty = 0$ on $U \cap F_\infty^{-1}(0)$, we get a continuous function on U which is holomorphic outside $F_\infty^{-1}(0)$. Moreover, we have

$$|w_\infty| \;\leq\; \bar C \, F_\infty. \tag{4.8}$$

Let $\eta : \mathbb{R} \mapsto [0,1]$ be a cut-off function satisfying: $\eta(t) = 0$ if $t \leq 1$, $\eta(t) = 1$ if $t \geq 2$ and $|\eta'| \leq 1$. Then for any smooth function φ with closure of its support contained in U, we have

$$\int_U w_\infty \, \bar\partial \varphi \, \omega_\infty^n \;=\; \lim_{\epsilon \to 0} \int_U w_\infty \, \eta(\epsilon^{-1} F_\infty) \, \bar\partial \varphi \, \omega_\infty^n. \tag{4.9}$$

Since F_∞ is a Lipschtz function, by using (4.8) and integration by parts, the right-handed side of (4.9) is bounded by a constant multiple of

$$\lim_{\epsilon \to 0} \int_{U \cap \{F_\infty \leq 2\epsilon\}} \omega_\infty^n \;=\; 0.$$

[16] The singular set $\mathcal{S}$ may overlap with $F_\infty^{-1}(0)$ along a subset.

Therefore, in the sense of distribution, we have

$$\bar{\partial} w_\infty = 0 \quad \text{in } U.$$

Thus by the standard elliptic theory, being continuous, w_∞ extends to be a holomorphic function on U, consequently, σ_∞ extends to be a holomorphic section of $K_{M_\infty}^{-\lambda}$ and $F_\infty = ||\sigma_\infty||_\infty$ outside $\mathcal{S}$.

Next we exam the limit of D under the convergence of (M, ω_i). Since $||\sigma_i||_i = 0$ on D, the limit of D must lie in D_∞, where D_∞ denotes the zero set of F_∞. We claim that the limit of D coincides with D_∞. If this is not true, there are $x \in D_\infty$ and $r > 0$ such that $B_{2r}(x, d_\infty) \cap D_\infty$ is disjoint from the limit of D. Choose $x_i \in M$ going to x as (M, ω_i) converge to (M_∞, d_∞), then for i sufficiently large, $B_r(x_i, \omega_i)$ is disjoint from D, so lies in the smooth part of (M, ω_i). The regularity theory in [CCT02] implies that $\mathcal{S} \cap B_r(x, d_\infty)$ is of complex codimension at least 2 and near a generic point $y \in B_r(x, d_\infty) \cap D_\infty$, σ_∞ is holomorphic and defines D_∞, moreover, the convergence of (M, ω_i) to (M_∞, d_∞) is in C^∞-topology and σ_i converge to σ_∞ near y , so σ_i must vanish somewhere in $B_r(x_i, \omega_i)$, a contradiction. This shows that the limit of D coincides with D_∞.

If $\beta_\infty = 1$, the singular set $\mathcal{S}$ is of complex dimension at least 2 and $\sigma_\infty \in H^0(M_\infty, K_{M_\infty}^{-\lambda})$ which consists of all holomorphic sections of $K_{M_\infty}^{-\lambda}$ on $M_\infty \backslash \mathcal{S}$. Then D_∞ is simply the divisor $\{\sigma_\infty = 0\}$.

Summarizing the above discussions, we have

Theorem 4.3. *Let $(M_\infty, \omega_\infty)$, $\mathcal{S}$ etc. be as in Theorem 3.1. Then (M, ω_i) converge to $(M_\infty, \omega_\infty)$ in the C^∞-topology outside a closed subset $\bar{\mathcal{S}} \cup D_\infty$, where $\bar{\mathcal{S}}$ (possibly empty) is of codimension at least 4, and D converges to D_∞ in the Gromov-Hausdorff topology. If $\beta_\infty < 1$, $\mathcal{S} = \bar{\mathcal{S}} \cup D_\infty$. If $\beta_\infty = 1$, $\mathcal{S} = \bar{\mathcal{S}}$ and D_∞ is a divisor of $K_{M_\infty}^{-\lambda}$.*[17]

Remark 4.4. *As an easy consequence of this theorem, we can also get the smooth convergence to tangent cones:* Let $\mathcal{C}_x$ be a tangent cone which is the limit of $(M_\infty, r_i^{-2}\omega_\infty, x)$, then $(M_\infty, r_i^{-2}\omega_\infty, x)$ converge to $\mathcal{C}_x$ in the C^∞-topology outside its singular set $\mathcal{S}_x$.

This can be seen as follows: If $\beta_\infty < 1$, this is already clear in Section 3, or more precisely, this follows from the proof of Theorem 3.1. If $\beta_\infty = 1$, for any $y \in \mathcal{C}_x \backslash \mathcal{S}_x$[18] *and r sufficiently small, we have*

$$\mathrm{Vol}(B_r(y, g_x)) \geq (c(n) - \epsilon)\, r^{2n},$$

where ϵ is chosen to be small so that for any $\bar{y} \in \mathcal{S}$,

$$\mathrm{Vol}(B_r(\bar{y}, \omega_\infty)) \leq (c(n) - 2\epsilon)\, r^{2n}.$$

[17] It follows from the partial C^0-estimate in the next section that the same holds even if $\beta_\infty < 1$.

[18] By [TW12] or Appendix 2, $\mathcal{S}_x$ is of complex codimension at least 2.

Let $y_i \in M_\infty$ with $\lim y_i = y$, then by the same arguments as those in the proof of Theorem 3.1, for some $N = N(\epsilon)$ and $\bar r = r/N$, the ball $B_{\bar r}(y_i, r_i^{-2}\omega_\infty)$ lies entirely in the regular part of M_∞, then the smooth convergence follows from a result of Anderson [An90].

5 Partial C^0-estimate

In this section, we prove Theorem 1.2. By our results on compactness of conic Kähler-Einstein metrics in last two sections, we need to prove only the following:

Theorem 5.1. *Let M be a Fano manifold M and D be a smooth divisor whose Poincaré dual is $\lambda\, c_1(M)$. Let ω_i be a sequence of conic Kähler-Einstein metrics on M with cone angle $2\pi\beta_i$ along D satisfying:*

$$\lim \beta_i \;=\; \beta_\infty > 0 \quad \text{and} \quad 1 - (1-\beta_\infty)\,\lambda \;>\; 0.$$

We also assume that (M, ω_i) converge to a (possibly singular) conic Kähler-Einstein manifold $(M_\infty, \omega_\infty)$ as described in Theorem 4.3. Then there are uniform constants $c_k = c(k, n, \lambda, \beta_\infty) > 0$ for $k \geq 1$ and $\ell_a \to \infty$ such that for $\ell = \ell_a$,

$$\rho_{\omega_i,\ell} \;\geq\; c_\ell \;>\; 0. \tag{5.1}$$

For the readers' convenience, we recall the definition of $\rho_{\omega_i,\ell}$ as well as a few facts.

Let H_i be the Hermitian metric on K_M^{-1} associated to ω_i, then we have an induced inner product $< \cdot, \cdot >_i$ on each $H^0(M, K_M^{-\ell})$ as follows:

$$< S, S' >_i = \int_M H_i^\ell(S, S')\,\omega_i^n, \qquad \forall\, S, S' \in H^0(M, K_M^{-\ell}).$$

Let $\{S_\alpha\}_{0\leq\alpha\leq N}$ be any orthonormal basis of $H^0(M, K_M^{-\ell})$ with respect to the inner product $< \cdot, \cdot >_i$, then we have

$$\rho_{\omega_i,\ell}(x) \;=\; \sum_{\alpha=0}^{N} H_i(S_\alpha, S_\alpha)(x), \tag{5.2}$$

We have shown in last section that the defining sections σ_i of D normalized with respect to H_i converge to a holomorphic section σ_∞ of $K_{M_\infty}^{-\lambda}$ on either $M_\infty \backslash \mathcal{S}$ for $\beta_\infty < 1$ or $M_\infty \backslash \mathcal{S} \cup D_\infty$ for $\beta_\infty = 1$, satisfying: In any local coordinates $z_1, \cdots, z_n$ outside $\mathcal{S}$, we have

$$(\det(g_{a\bar b}))^\lambda\, |w|^2 \;<\; \infty \tag{5.3}$$

where

$$\sigma_\infty \;=\; w\left(\frac{\partial}{\partial z_1} \wedge \cdots \wedge \frac{\partial}{\partial z_n}\right)^{\otimes \lambda} \quad \text{and} \quad \omega_\infty \;=\; \sqrt{-1}\sum_{a,b=1}^{n} g_{a\bar b}\, dz_a \wedge d\bar z_b\,.$$

Define a Hermitian metric H_∞ on $K_{M_\infty}^{-1}$ on $M_\infty \backslash \mathcal{S}$ by

$$H_\infty = \tilde{H}_\infty(\sigma_\infty, \sigma_\infty)^{\frac{1-\beta}{\mu}} \tilde{H}_\infty. \tag{5.4}$$

Here $\tilde{H}_\infty$ denotes the Hermitian metric induced by the determinant of ω_∞. The following can be easily proved.

Lemma 5.2. *The Hermitian metrics H_i converge to H_∞ on $M_\infty \backslash \mathcal{S}$ in the C^∞-topology. Moreover, we have*

$$H_\infty(\sigma_\infty, \sigma_\infty) < \infty \quad \text{and} \quad \int_{M_\infty} H_\infty(\sigma_\infty, \sigma_\infty)\, \omega_\infty^n = 1.$$

By a holomorphic section of $K_{M_\infty}^{-\ell}$ on M_∞ ($\ell > 0$), we mean a holomorphic section σ of $K_{M_\infty}^{-\ell}$ on $M_\infty \backslash \mathcal{S}$ with $H_\infty(\sigma, \sigma)$ bounded. We denote by $H^0(M_\infty, K_{M_\infty}^{-\ell})$ the space of all holomorphic sections of $K_{M_\infty}^{-\ell}$ on M. If M_∞ is smooth outside a closed subset of codimension 4, then it coincides with the definition we used in literature.

Applying Corollary 4.2 and standard arguments, we can prove:

Lemma 5.3. *For any fixed $\ell > 0$, if $\{\tau_i\}$ is any sequence of $H^0(M, K_M^{-\ell})$ satisfying:*

$$\int_M H_i(\tau_i, \tau_i)\, \omega_i^n = 1,$$

then a subsequence of τ_i converges to a section τ_∞ in $H^0(M_\infty, K_{M_\infty}^{-\ell})$.

Furthermore, since $\rho_{\omega_i,\ell}$ are uniformly continuous, it follows from Lemma 5.3 that a subsequence of them converges to a continuous function on M_∞.[19]

We note that if (5.1) holds for ℓ, so does for ℓ^k for any $k \geq 2$. This can be easily verified by using the definition of $\rho_{\omega_i,\ell}$ and Corollary 4.2. Therefore, in order to prove Theorem 5.1, we only need to show that there is an ℓ such that,

$$\inf_i \inf_{x \in M} \rho_{\omega_i,\ell}(x) > 0. \tag{5.5}$$

Next we claim that (5.5) follows from the following: For any $x \in M_\infty$, there is an $\ell = \ell_x$ and a sequence $x_i \in M$ such that $\lim x_i = x$ and

$$\inf_i \rho_{\omega_i,\ell}(x_i) > 0. \tag{5.6}$$

For the readers' convenience, we show how to derive (5.5) from this claim: Given any $x \in M$, by using the estimate in Corollary 4.2 and (5.6), there is a $r = r_x$ such that

$$\inf_i \inf_{B_r(x_i,\omega_i)} \rho_{\omega_i,\ell} > 0.$$

Since M_∞ is compact, there are finitely many $x = x_a$, $\ell = \ell_a$ and $r = r_a$ as above ($a = 1, \cdots, k$) such that the balls $B_{r_a}(x_a, \omega_\infty)$ cover M_∞. Then (5.5) holds for $\ell = \ell_1 \cdots \ell_k$. Hence, it suffices to prove (5.6).

[19] In fact, the limit is equal to $\rho_{\omega_\infty,\ell}$ as shown in the end of this section.

The following lemma provides the L^2-estimate for $\bar\partial$-operator on (M, ω_i). It can be proved by using the smooth approximations of ω_i constructed in Theorem 2.6.

Lemma 5.4. *For any $\ell > 0$, if ζ is a (0,1)-form with values in $K_M^{-\ell}$ and $\bar\partial\zeta = 0$, there is a smooth section ϑ of $K_M^{-\ell}$ such that $\bar\partial\vartheta = \zeta$ and*

$$\int_M ||\vartheta||_i^2 \,\omega_i^n \,\le\, \frac{1}{\ell+\mu}\int_M ||\zeta||_i^2\,\omega_i^n,$$

where $||\cdot||_i$ denotes the norm induced by H_i and ω_i.

We have seen that for any $r_j \mapsto 0$, by taking a subsequence if necessary, we have a tangent cone $\mathcal{C}_x$ of $(M_\infty, \omega_\infty)$ at x, which is the limit of $(M_\infty, r_j^{-2}\omega_\infty, x)$ in the Gromov-Hausdorff topology, satisfying:

$\mathbf{T}_1$. Each $\mathcal{C}_x$ is regular outside a closed subcone $\mathcal{S}_x$ of complex codimension at least 1. Such a $\mathcal{S}_x$ is the singular set of $\mathcal{C}_x$;

$\mathbf{T}_2$. There is an natural Kähler Ricci-flat metric g_x on $\mathcal{C}_x\backslash\mathcal{S}_x$ which is also a cone metric. Its Kähler form ω_x is equal to $\sqrt{-1}\,\partial\bar\partial\rho_x^2$ on the regular part of $\mathcal{C}_x$, where ρ_x denotes the distance function from the vertex of $\mathcal{C}_x$, denoted by o.

We will denote by L_x the trivial bundle $\mathcal{C}_x \times \mathbb{C}$ over $\mathcal{C}_x$ equipped with the Hermitian metric $e^{-\rho_x^2}\,|\cdot|^2$. The curvature of this Hermitian metric is given by ω_x.

Recall that $\mathcal{S}_k$ $(k = 0, 1, \cdots, 2n-1)$ consists of points in M_∞ for which no tangent cone splits off a factor, $\mathbb{R}^{k+1}$, isometrically. Then it was shown in [CC95]

$$\mathcal{S}_{2n-1} = \emptyset, \quad \mathcal{S}_0 \subset \cdots \subset \mathcal{S}_{2n-2} = \mathcal{S} \quad\text{and}\quad \dim\mathcal{S}_k \le k.$$

The following lemma is a consequence of Theorem 3.2 and [CCT02].

Lemma 5.5. *We have For any $x \in \mathcal{S}_{2n-2}\backslash\mathcal{S}_{2n-4}$, we have*

(1) $\mathcal{S}_{2k+1} = \mathcal{S}_{2k}$ for $k = 0, \cdots, n-1$ and if $x \in \mathcal{S}_{2n-2}$, then $\mathcal{C}_x = \mathbb{C}^{n-1}\times\mathcal{C}'_x$ and g_x is a product of the Euclidean metric on $\mathbb{C}^{n-1}$ with a flat conic metric on $\mathcal{C}'_x$ of angle $2\pi\bar\beta$;

(2) If $x \in \mathcal{S}_{2n-4}$ and $\mathcal{S}_x$ is of complex codimension 1, then there is a subcone $\bar{\mathcal{S}}_x \subset \mathcal{S}_x$ of complex codimension at least 2 such that a tangent cone of $(\mathcal{C}_x, g_x)$ at y is isometric to a product of the Euclidean metric on $\mathbb{C}^{n-1}$ with a flat conic metric on $\mathcal{C}'_x$ of angle $2\pi\bar\beta$;

(3) There is a $\bar\beta_\infty$ depending only on the diameter and volume of $(M_\infty, \omega_\infty)$ such that $\bar\beta_\infty \le \bar\beta \le \beta_\infty$ for $\bar\beta$ in (1) and (2);

(4) If $\beta_\infty < 1$, then $(1-\bar\beta) = m(1-\beta_\infty)$ for $\bar\beta$ in (1) and (2), in particular, there are only finitely many such $\bar\beta$'s.

Proof. (1) and (2) follow from Theorem 9.1 in [CCT02]. (3) follows from the Bishop-Gromov volume comparison. (4) follows from Theorem 3.2, **C4**. ☐

Remark 5.6. *Using a local version of the partial C^0-estimate (cf. Appendix 1), one can actually prove that $\bar{\mathcal{S}}_x$ is closed. But we do not need this property in proving the partial C^0-estimate, and consequently, Theorem 1.1.*

Now we fix some notations: For any $\epsilon > 0$, we put

$$V(x;\epsilon) = \{ y \in \mathcal{C}_x \,|\, y \in B_{\epsilon^{-1}}(0,g_x) \setminus \overline{B_\epsilon(0,g_x)},\ d(y,\mathcal{S}_x) > \epsilon \},$$

where $B_R(o,g_x)$ denotes the geodesic ball of $(\mathcal{C}_x,g_x)$ centered at the vertex and with radius R.

If $\mathcal{C}_x$ has isolated singularity, then $\mathcal{S}_x = \{o\}$ and

$$V(x;\epsilon) = \{ y \in \mathcal{C}_x \,|\, y \in B_{\epsilon^{-1}}(0,g_x) \setminus \overline{B_\epsilon(0,g_x)} \}.$$

Let $\{r_j\}$ be any sequence such that r_j^{-2} are integers and $(M_\infty, r_j^{-2}\omega_\infty, x)$ converges to $(\mathcal{C}_x, g_x, o)$. By [CCT02], for any $\epsilon > 0$ and $\delta > 0$, we can have a $j_0 = j_0(\epsilon,\delta)$ such that for each $j \ge j_0$, there is a diffeomorphisms $\phi : V(x;\frac{\epsilon}{4}) \mapsto M_\infty \backslash \mathcal{S}$, where $\mathcal{S}$ is the singular set of M_∞, satisfying:

(1) $d(x,\phi(V(x;\epsilon))) < 10\,\epsilon r$ and $\phi(V(x;\epsilon)) \subset B_{(1+\epsilon^{-1})r}(x)$, where $r = r_j$ and $B_R(x)$ denotes the geodesic ball of (M_∞,ω_∞) with radius R and center at x;

(2) If g_∞ is the Kähler metric with the Kähler form ω_∞ on $M_\infty \backslash \mathcal{S}$, then

$$||r^{-2}\phi^* g_\infty - g_x||_{C^6(V(x;\frac{\epsilon}{2}))} \le \delta, \tag{5.7}$$

where the norm is defined in terms of the metric g_x.

Lemma 5.7. *Given $\epsilon > 0$ and any sufficiently small $\delta > 0$, there are a sufficiently large $\ell = r^{-2}$, a diffeomorphism $\phi : V(x;\frac{\epsilon}{4}) \mapsto M_\infty \backslash \mathcal{S}$ with properties (1) and (2) above, and an isomorphism ψ from the trivial bundle $\mathcal{C}_x \times \mathbb{C}$ onto $K_{M_\infty}^{-\ell}$ over $V(x;\epsilon)$ commuting with ϕ satisfying:*

$$||\psi(1)||^2 = e^{-\rho_x^2} \quad \text{and} \quad ||\nabla\psi||_{C^4(V(x;\epsilon))} \le \delta, \tag{5.8}$$

where $||\cdot||$ denotes the induced norm on $K_{M_\infty}^{-\ell}$ by ω_∞, ∇ denotes the covariant derivative with respect to the metrics $||\cdot||^2$ and $e^{-\rho_x^2}|\cdot|^2$.

Proof. The arguments are pretty standard, so we just give an outlined proof.

Let $\{r_j\}$ be as above such that $(M_\infty, r_j^{-2}\omega_\infty, x)$ converges to $(\mathcal{C}_x, g_x, o)$. Assume $\epsilon' < \epsilon$ determined later. Then there are diffeomorphisms $\phi_j : V(x;\frac{\epsilon'}{4}) \mapsto M_\infty \backslash \mathcal{S}$ satisfying:

$$d(x,\phi_j(V(x;\epsilon'))) < 10\,\epsilon' r_j, \quad \phi_j(V(x;\epsilon')) \subset B_{(1+\epsilon'^{-1})r_j}(x)$$

and

$$\lim_{j\to\infty} ||r_j^{-2}\phi_j^* g_\infty - g_x||_{C^6(V(x;\frac{\epsilon'}{2}))} = 0.$$

We cover $V(x;\epsilon')$ by finitely many geodesic balls $B_{s_\alpha}(y_\alpha)$ $(1 \le \alpha \le N)$ satisfying:

(i) The closure of each $B_{2s_\alpha}(y_\alpha)$ is strongly convex and contained in $\mathcal{C}_x\backslash\mathcal{S}_x$;.

(ii) The half balls $B_{s_\alpha/2}(y_\alpha)$ are mutually disjoint;

(iii) $s_\alpha \,\geq\, \nu_x\, d(y_\alpha, \mathcal{S}_x)$, where ν_x is a constant depending only on $V(x,\epsilon)$.[20]

We will first set $\ell \,=\, \ell_j = r_j^{-2}$ and $\phi \,=\, \phi_j$ when j is sufficiently large and construct ψ.

First we construct $\tilde\psi_\alpha$ over each $B_{2s_\alpha}(y_\alpha)$. For any $y \in B_{2s_\alpha}(y_\alpha)$, let $\gamma_y \subset B_{2s_\alpha}(y_\alpha)$ be the unique minimizing geodesic from y_α to y. We define $\tilde\psi_\alpha$ as follows: First we define $\tilde\psi_\alpha(1) \in L|_{\phi(y_\alpha)}$ such that

$$||\tilde\psi_\alpha(1)||^2 \,=\, e^{-\rho_x^2(y_\alpha)},$$

where $L \,=\, K_{M_\infty}^{-\ell}$. Next, for any $y \in U_\alpha$, where $U_\alpha \,=\, B_{2s_\alpha}(y_\alpha)$, define

$$\tilde\psi_\alpha : \mathbb{C} \mapsto L|_y, \quad \tilde\psi_\alpha(a(y)) = \tau(\phi(y)),$$

where $a(y)$ is the parallel transport of 1 along γ_y with respect to the norm $e^{-\rho_x^2}\,|\cdot|^2$ and $\tau(\phi(y))$ is the parallel transport of $\psi(1)$ along $\phi\circ\gamma_y$ with respect to the norm $||\cdot||^2$.

Clearly, we have the first equation in (5.8). The estimates on derivatives can be done as follows: If $a : U_\alpha \mapsto U_\alpha \times \mathbb{C}$ and $\tau : U_\alpha \mapsto \phi^* L|_{U_\alpha}$ are two sections such that $\tilde\psi_\alpha(a) \,=\, \tau$, then we have the identity:

$$\nabla\tau \,=\, \nabla\tilde\psi_\alpha(a) \,+\, \tilde\psi_\alpha(\nabla a),$$

where ∇ denote the covariant derivatives with respect to the given norms on line bundles $\mathcal{C}_x \times \mathbb{C}$ and L. By the definition of $\tilde\psi_\alpha$, one can easily see that $\nabla\tilde\psi_\alpha(y_\alpha) \equiv 0$. To estimate $\nabla\tilde\psi_\alpha$ at y, we differentiate along γ_y to get

$$\nabla_T\nabla_X\tau \,=\, \nabla_T(\nabla_X\tilde\psi_\alpha(a)) + \tilde\psi_\alpha(\nabla_T\nabla_X a),$$

where T is the unit tangent of γ_y and X is a vector field along γ_y with $[T,X] = 0$. Here we have used the fact that $\nabla_T\tilde\psi_\alpha = 0$ which follows from the definition of $\tilde\psi_\alpha$. Using the curvature formula and the fact that a is parallel along γ_y, we see that it is the same as

$$\ell\,\phi^*\omega_\infty(T,X)\,\tilde\psi_\alpha(a) \,=\, \nabla_T(\nabla_X\tilde\psi_\alpha(a)) \,+\, \omega_x(T,X)\,a.$$

Using the fact that ω_x is the limit of $i\phi^*\omega_\infty$ as i tends to ∞, we can deduce from the above that $\nabla_T(\nabla_X\tilde\psi_\alpha(a))$ is sufficiently small so long as $\ell = \ell_j$ is sufficiently large. Since $\nabla_X\tilde\psi_\alpha = 0$ at y_α, we see that $||\nabla\tilde\psi_\alpha||_{C^0(U_\alpha)}$ can be made sufficiently small. The higher derivatives, say up to order 6, can be bounded inductively in a similar way.

[20] Property (iii) is not needed in the subsequent proof. For proving this lemma, we may simply take a cover of $V(x;\epsilon')$ by balls of comparable size such that (i) holds and choose j sufficiently large.

Next we want to modify each $\tilde{\psi}_\alpha$. For any α, β, we set

$$\theta_{\alpha\gamma} = \tilde{\psi}_\alpha^{-1} \circ \tilde{\psi}_\gamma : U_\alpha \cap U_\gamma \mapsto S^1.$$

Clearly, we have

$$\theta_{\alpha\kappa} = \theta_{\alpha\gamma} \cdot \theta_{\gamma\kappa} \quad \text{on} \quad U_\alpha \cap U_\gamma \cap U_\kappa,$$

so we have a closed cycle $\{\theta_{\alpha\gamma}\}$. By the derivative estimates on each $\tilde{\psi}_\alpha$, we know that each $\theta_{\alpha\gamma}$ is close to a constant. Therefore, we can modify $\tilde{\psi}_\alpha$'s such that each transition function $\theta_{\alpha\gamma}$ is a unit constant, that is, we can construct $\zeta_\alpha : U_\alpha \mapsto S^1$ such that if we replace each $\tilde{\psi}_\alpha$ by $\tilde{\psi}_\alpha \cdot \zeta_\alpha$, the corresponding transition functions are constant. Moreover we can dominate $||\nabla \zeta_\alpha||_{C^4}$ by the norm $||\nabla \tilde{\psi}_\alpha||_{C^5}$ (possibly) on a slightly larger ball.

The cycle $\{\theta_{\alpha\gamma}\}$ of constants gives rise to a flat bundle F, and we have constructed an isomorphism

$$\xi : F \mapsto K_{M_\infty}^{-\ell}$$

over an neighborhood of $\overline{V(x;\epsilon')}$ satisfying all the estimates in (5.8).

If we replace ℓ by $k\ell$, we get an analogous isomorphism

$$\xi^k : F^k \mapsto K_{M_\infty}^{-k\ell}.$$

We want to choose k to get the required ℓ, ϕ and ψ. Set

$$U(x;\epsilon',\epsilon) = \{\, y \in \mathcal{C}_x \mid \sqrt{\epsilon'} < \rho_x(y) < \epsilon^{-1},\ \bar{y} \in E_\epsilon \,\},$$

where $y = \rho_x(y)\,\bar{y}$ and $E_\epsilon \subset \partial B_1(o, g_x) \backslash \mathcal{S}_x$ is an open submanifold containing all $\bar{z} \in \partial B_1(o, g_x))$ with

$$\bar{d}(\bar{z}, \mathcal{S}_x \cap \partial B_1(o, g_x)) \geq \epsilon^2,$$

where $\bar{d}(\cdot,\cdot)$ denotes the distance function on $\partial B_1(o, g_x)$. Furthermore, we can choose E_ϵ such that its topology depends only on ϵ and $\mathcal{S}_x$.

Assume that ϵ' is sufficiently such that $U(x;\epsilon',\epsilon) \subset V(x;\epsilon')$.

Since the flat bundle $F|_{U(x;\epsilon',\epsilon)}$ is given by a representation

$$\rho : \pi_1(U(x;\epsilon',\epsilon)) = \pi_1(E_\epsilon) \mapsto S^1.$$

Note that ρ is the pull-back of a homomorphism $\bar{\rho} : H_1(E_\epsilon, \mathbb{Z}) \mapsto S^1$ through the natural projection: $\pi_1(E_\epsilon) \mapsto H_1(E_\epsilon, \mathbb{Z})$. Clearly, $H_1(E_\epsilon, \mathbb{Z})$ is the sum of an abelian group of finite rank m and a finite group of order ν. Observing that m and ν depend only on ϵ and $\mathcal{S}_x$, there is an k, which may depend on m, ν and δ, such that F^k is essentially trivial on the scale of δ, i.e., the corresponding transition functions are in a δ'-neighborhood of the identity in S^1, where $\delta' s$ depends only on and much smaller than δ.

We reset ℓ to be the k-multiple of the initial ℓ. If ϵ' is much smaller than ϵ and k^{-1}, we have

$$k^{-\frac{1}{2}} V(x;\epsilon) := \{\, y \in \mathcal{C}_x \mid \epsilon < \sqrt{k}\,\rho_x(y) < \epsilon^{-1},\ \sqrt{k}\, d(y, \mathcal{S}_x) > \epsilon \,\} \subset U(x;\epsilon',\epsilon).$$

We can redefine ϕ as the composition of the scaling map $y \mapsto k^{-1}y : \mathcal{C}_x \mapsto \mathcal{C}_x$ and the initial ϕ. Then this newer ϕ maps $V(x;\epsilon)$ onto $k^{-\frac{1}{2}}V(x;\epsilon)$. Since $(M_\infty, kr_j^{-2}\omega_\infty, x)$ still converge to the cone $(\mathcal{C}_x, g_x, x)$, this newer ϕ satisfies properties (1) and (2) required by our above discussions if j is sufficiently large. Thus, we can apply the above for this newer ϕ to get corresponding $\tilde{\psi}_\alpha$, F etc.. The newer flat bundle F, which is the same as F^k for older ϕ, has transition functions in a δ'-neighborhood of the identity in S^1

By modifying $\tilde{\psi}_\alpha$ restricted to E_ϵ, we can construct a bundle isomorphism $\zeta : \mathcal{C}_x \times \mathbb{C} \mapsto F$ over E_ϵ whose norm is bounded by a constant much smaller than δ. This ζ extends trivially to an isomorphism over $U(x; k^{-1/2}\epsilon^2, k^{1/2}\epsilon)$ with controlled norm. It follows that $\psi = \xi^k \cdot \zeta$ gives the required isomorphism between $\mathcal{C}_x \times \mathbb{C}$ and $K_{M_\infty}^{-\ell}$ over $\phi(V(x;\epsilon))$. This completes a proof of this lemma. □

In the above lemma, k may depend on x, or more precisely, $\mathcal{C}_x$. There is another approach to choosing k which depends only on n and β_∞. The key is to show that $\mathcal{C}_x \backslash \bar{\mathcal{S}}_x$ has finite fundamental group of order $\nu \geq 1$ which depends only on n. Then we just need to take ℓ to be a multiple of ν such that $\ell\beta_\infty$ is sufficiently close to 1 modulo $\mathbb{Z}$.

Now we prove (5.6), consequently, the partial C^0-estimate for conic Kähler-Einstein metrics. As for smooth Kähler-Einstein metrics, we will apply the L^2-estimate to proving (5.6). The method is standard now and resembles the one we used for Del-Pezzo surfaces in [Ti90]. First we construct an approximated holomorphic section $\tilde{\tau}$ on M_∞, then one can perturb it into a holomorphic section τ by the L^2-estimate for $\bar{\partial}$-operators, finally, one uses the derivative estimate in Corollary 4.2 to conclude that $\tau(x) \neq 0$. These steps are similar to those used in [DS14] as well as [Ti13] in establishing the partial C^0-estimate for smooth Kähler-Einstein metrics.

Let $\epsilon > 0$ and $\delta > 0$ be sufficiently small and be determined later. We fix $\ell = r^{-2}$, where $r = r_j$ for a sufficiently large j, such that Lemma 5.7 holds for ℓ, ϵ and δ. Choose ϕ and ψ by Lemma 5.7, then there is a section $\tau = \psi(1)$ of $K_{M_\infty}^{-\ell}$ on $\phi(V(x;\epsilon))$ satisfying:

$$||\tau||^2 = e^{-\rho_x^2}. \tag{5.9}$$

By Lemma 5.7, for some uniform constant C, we have

$$||\bar{\partial}\tau|| \leq C\,\delta. \tag{5.10}$$

Now let us state a technical lemma.

Lemma 5.8. *For any $\bar{\epsilon} > 0$, there is a smooth function $\gamma_{\bar{\epsilon}}$ on $\mathcal{C}_x$ satisfying:*

(1) $\gamma_{\bar{\epsilon}}(y) = 1$ if $d(y, \mathcal{S}_x) \geq \bar{\epsilon}$, where $d(\cdot,\cdot)$ is the distance of $(\mathcal{C}_x, g_x)$;

(2) $0 \leq \gamma_{\bar{\epsilon}} \leq 1$ and $\gamma_{\bar{\epsilon}}(y) = 0$ in an neighborhood of $\mathcal{S}_x$;

(3) $|\nabla \gamma_{\bar\epsilon}| \le C$ *for some constant* $C = C(\bar\epsilon)$ *and*

$$\int_{B_{\bar\epsilon^{-1}}(o,g_x)} |\nabla \gamma_{\bar\epsilon}|^2\, \omega_x^n \le \bar\epsilon.$$

Proof. This is rather standard and has been known to me for quite a while. The arguments are based on known techniques: First we prove this lemma in a simple case, then we reduce the general case to this case by using partial C^0-estimate already established. But the arguments are tedious and lengthy, so we will refer the readers to Appendix 1 for its complete proof. Here we only prove this lemma in a simple case and explains briefly why Lemma 5.8 should be true in general.

A key reason is the fact that the Poincaré metric on a punctured disc has finite volume.

Consider the simple case: $\mathcal{S}_x = \mathbb{C}^{n-1}$, i.e., $\mathcal{C}_x$ is of the form $\mathbb{C}^{n-1} \times \mathcal{C}'_x$, where $\mathcal{C}'_x$ is biholomorphic to $\mathbb{C}$, moreover, the cone metric g_x coincides with the standard cone metric

$$g_{\bar\beta} = \sum_{i=1}^{n-1} dz_i d\bar z_i + (d\rho^2 + \bar\beta^2 \rho^2 d\theta^2),$$

where $z_1, \cdots, z_{n-1}$ are coordinates of $\mathbb{C}^{n-1}$ and $0 < \bar\beta < \beta_\infty$, e.g., one of them in Lemma 5.5, (3) or (4). Clearly, $\rho = d(y, \mathcal{S}_x)$.

We denote by η a cut-off function: $\mathbb{R} \mapsto \mathbb{R}$ satisfying: $0 \le \eta \le 1$, $|\eta'(t)| \le 1$ and

$$\eta(t) = 0 \ \text{ for } \ t > \log(-\log \bar\delta^3) \ \text{ and } \ \eta(t) = 1 \ \text{ for } \ t < \log(-\log \bar\delta).$$

Here $\bar\delta < 1/3$ is to be determined. Now we define as follows: If $\rho(y) \ge \bar\epsilon$, put $\gamma_{\bar\epsilon}(y) = 1$ and if $\rho(y) < \bar\epsilon$

$$\gamma_{\bar\epsilon}(y) = \eta\left(\log\left(-\log\left(\frac{\rho(y)}{\bar\epsilon}\right)\right)\right).$$

Clearly, $\gamma_{\bar\epsilon}$ is a smooth function and we have

$$\gamma_{\bar\epsilon}(y) = 1 \ \text{ if } \ \rho(y) \ge \bar\delta\bar\epsilon \quad \text{and} \quad \gamma_{\bar\epsilon}(y) = 0 \ \text{ if } \ \rho(y) \le \bar\delta^3\bar\epsilon.$$

Furthermore, the support of $|\nabla\gamma_{\bar\epsilon}|(y) = 0$ is contained in the region where $\bar\delta^3\bar\epsilon < \rho(y) < \bar\delta\bar\epsilon$. In the region, we have

$$|\nabla\gamma_{\bar\epsilon}| \le \frac{1}{\rho(-\log\frac{\rho}{\bar\epsilon})}.$$

It follows that

$$\int_{B_{\bar\epsilon^{-1}}(o,g_x)} |\nabla\gamma_{\bar\epsilon}|^2\, \omega_x^n \le \frac{a_{n-1}}{\bar\epsilon^{2n-2}} \int_{\bar\delta^3}^{\bar\delta} \frac{dr}{r(-\log r)^2} \le \frac{a_{n-1}}{\bar\epsilon^{2n-2}(-\log\bar\delta)},$$

where a_{n-1} denotes the volume of the unit ball in $\mathbb{R}^{2n-2}$.

Now choose $\bar\delta$ such that $a_{n-1} \le \bar\epsilon^{2n-1}(-\log\bar\delta)$, then we have

$$\int_{B_{\bar\epsilon^{-1}}(o,g_x)} |\nabla\gamma_{\bar\epsilon}|^2 \le \bar\epsilon.$$

Clearly, we also have $|\nabla\gamma_{\bar\epsilon}| \le C$ for some $C = C(\bar\epsilon)$.

In general, we know from Section 3 that $\mathcal{S}_x$ is a union of $\mathcal{S}_x^0$ and $\bar{\mathcal{S}}_x$, where $\bar{\mathcal{S}}_x$ is a subcone of complex codimension at least 2 and $\mathcal{S}_x^0$ consists of all $y \in \mathcal{S}_x$ such that a tangent cone of $(\mathcal{C}_x, g_x)$ at y is $\mathbb{C}^{n-1}\times\mathcal{C}'_y$ with the standard metric $g_{\bar\beta}$.

Using the fact that $\bar{\mathcal{S}}_x$ is of complex codimension at least 2, we can use standard methods to construct a function χ with required properties (1)-(3) in an neighborhood U of $\bar{\mathcal{S}}_x \cap B_{\epsilon^{-1}}(x,g_x)$ such that it is equal to 1 near ∂U. Then χ vanishes in an open neighborhood B and $\bar B \subset U$.

For each $y \in \mathcal{S}_x\backslash B$, there is a tangent cone $\mathcal{C}_y$ as that in the simple case we have considered above, so we can use the arguments in rest of this section to establish the partial C^0-estimate near y. This in turns gives some structure results for $\mathcal{C}_x$ near y and allows us to use the construction in the above simple case to get a function γ_y with required properties (1)-(3) on $B_{2r(y)}(y,g_x)$. Since $\mathcal{S}_x\backslash B$ is compact, we can cover it by finitely many balls $B_{r(y_b)}(y_b,g_x)$, then our γ will be obtained by using χ, those γ_{y_b}'s and a partition of unit associated to U and $B_{2r(y_b)}(y_b,g_x)$'s.

The detailed arguments along this line will be presented in Appendix 1. $\square$

Now, assuming Lemma 5.8, we continue the proof of the partial C^0-estimate.

First we define η to be a cut-off function satisfying:

$$\eta(t) = 1 \text{ for } t \le 1, \quad \eta(t) = 0 \text{ for } t \ge 2 \text{ and } |\eta'(t)| \le 1.$$

Let $\delta_0 > 0$ be determined later. Choose $\bar\epsilon$ such that $\gamma_{\bar\epsilon} = 1$ on $V(x;\delta_0)$. Then we choose ϵ such that $\delta > 4\,\epsilon$ and $V(x;\epsilon)$ contains the support of $\gamma_{\bar\epsilon}$ constructed in Lemma 5.8. Clearly, we can make $\bar\epsilon$ as small as we want if ϵ is sufficiently small.

We define for any $y \in V(x;\epsilon)$

$$\hat\tau(\phi(y)) = \eta\left(2\,\epsilon\left(\rho_x(y) + \rho_x(y)^{-1}\right)\right)\gamma_{\bar\epsilon}(y)\,\tau(\phi(y)). \tag{5.11}$$

It is easy to see that $\hat\tau$ vanishes near the boundary of $\phi(V(x;\epsilon))$, therefore, it extends to a smooth section of $K_{M_\infty}^{-\ell}$ on M_∞. Using that $\delta_0 > 4\epsilon$ and the definition of the cut-off function η, we deduce from (5.11)

$$\hat\tau = \tau \quad \text{on} \quad \phi(V(x;\delta_0)). \tag{5.12}$$

By a direct computation, we derive from (5.10) and (5.11)

$$\int_{M_\infty} ||\bar\partial\hat\tau||_\infty^2\,\omega_\infty^n \le \nu\, r^{2n-2}, \tag{5.13}$$

where $||\cdot||_\infty$ denotes the Hermitian norm associated to ω_∞ and $\nu = \nu(\delta,\epsilon)$, which can be made as small as we want so long as δ, ϵ and $\bar\epsilon$ are sufficiently small. Moreover, we have

$$\int_{M_\infty} ||\hat\tau||^2_\infty\,\omega^n_\infty \,\le\, \int_{\phi(V(x;\epsilon))} e^{-\rho_x^2}\,\omega^n_\infty \,<\,\infty. \tag{5.14}$$

Clearly, the support of $\hat\tau$ stays outside the singular set $\mathcal{S}$ of M_∞. We will modify $\hat\tau$ to get a newer section $\tilde\tau$ with support away from D_∞. If $\beta_\infty < 1$, then $\mathcal{S}$ contains D_∞ and consequently, we can take $\tilde\tau = \hat\tau$. If $\beta_\infty = 1$, D_∞ is a divisor defined by a holomorphic section σ_∞ of $K^{-\lambda}_{M_\infty}$. Put $\rho = ||\sigma_\infty||_\infty$. Let $\bar\eta$ be a cut-off function: $\mathbb{R}\mapsto\mathbb{R}$ satisfying: $0\le\bar\eta\le 1$, $|\bar\eta'(t)|\le 1$ and

$$\bar\eta(t) = 0 \;\text{ for }\; t > \log(-\log\hat\epsilon^2) \;\text{ and }\; \bar\eta(t) = 1 \;\text{ for }\; t < \log(-\log\hat\epsilon).$$

Now we define $\tilde\tau$ by

$$\tilde\tau(z) \,=\, \bar\eta\left(\log\left(-\log\rho(z)\right)\right)\,\hat\tau(z).$$

Then $\tilde\tau$ supports away from $\mathcal{S}\cup D_\infty$ and coincides with τ on $\{\rho\ge\hat\epsilon\}$. When $\hat\epsilon$ is sufficiently small, we can deduce from (5.13) and standard computations

$$\int_{M_\infty} ||\bar\partial\tilde\tau||^2_\infty\,\omega^n_\infty \,\le\, 2\,\nu\,r^{2n-2}. \tag{5.15}$$

Of course, this is automatically true if $\beta_\infty < 1$.

Set $U(x;\epsilon)$ to be $\phi(V(x;\epsilon))$ if $\beta_\infty<1$ and $\phi(V(x;\epsilon))\backslash\{z\,|\,d_\infty(z,D_\infty)\le\epsilon\}$ if $\beta_\infty = 1$, where $d_\infty(\cdot,D_\infty)$ denotes the distance from D_∞ with respect to ω_∞. We choose $\hat\epsilon$ such that $\tilde\tau=\tau$ on $U(x;\delta_0)$. Clearly, the support of $\tilde\tau$ is contained in $U(x;\epsilon)$ if ϵ is sufficiently small.

Note that $(M\backslash D,\omega_i)$ converge to $(M_\infty\backslash\mathcal{S}\cup D_\infty,\omega_\infty)$ and the Hermitian metrics H_i on K_M^{-1} converge to H_∞ on $M_\infty\backslash(\mathcal{S}\cup D_\infty)$ in the C^∞-topology. Therefore, for a sequence $\delta_i>0$ with $\lim\delta_i = 0$, there are diffeomorphisms

$$\tilde\phi_i : M_\infty\backslash T_i\left(\mathcal{S}\cup D_\infty\right) \,\mapsto\, M\backslash T_i(D)$$

and smooth isomorphisms

$$F_i : K^{-\ell}_{M_\infty} \,\mapsto\, K^{-\ell}_M$$

over $M_\infty\backslash T_i\left(\mathcal{S}\cup D_\infty\right)$, where

$$T_i(D) = \{x\in M \mid d_i(x,D)\,\le\,\delta_i\}$$

and

$$T_i\left(\mathcal{S}\cup D_\infty\right) = \{x\in M_\infty \mid d_\infty(x,\mathcal{S}\cup D_\infty)\,\le\,\delta_i\},$$

where $d_i(\cdot,D)$ (resp. $d_\infty(\cdot,\mathcal{S}\cup D_\infty)$) denotes the distance from D with respect to the metric ω_i (resp. ω_∞), satisfying:

$\mathbf{C}_1$: $\tilde{\phi}_i(M_\infty \backslash T_i(\mathcal{S} \cup D_\infty)) \subset M \backslash T_i(D)$;

$\mathbf{C}_2$: $\pi_i \circ F_i = \tilde{\phi}_i \circ \pi_\infty$, where π_i and π_∞ are corresponding projections;

$\mathbf{C}_3$: $||\tilde{\phi}_i^* \omega_i - \omega_\infty||_{C^2(M_\infty \backslash T_i(\mathcal{S} \cup D_\infty))} \leq \delta_i$;

$\mathbf{C}_4$: $||F_i^* H_i - H_\infty||_{C^4(M_\infty \backslash T_i(\mathcal{S} \cup D_\infty))} \leq \delta_i$.

We may assume i sufficiently large so that $U(x;\epsilon) \subset M_\infty \backslash T_i(\mathcal{S} \cup D_\infty)$.

Put $\tilde{\tau}_i = F_i(\tilde{\tau})$, then it follows from the definition of $\tilde{\tau}$

$$\tilde{\tau}_i = F_i(\tau) \quad \text{on} \quad \tilde{\phi}_i(U(x;\delta_0)). \tag{5.16}$$

Because of (5.15), for i sufficiently large, we have

$$\int_{M_i} ||\bar{\partial} \tilde{\tau}_i||_i^2 \, \omega_i^n \leq 3 \, \nu \, r^{2n-2}, \tag{5.17}$$

where $|| \cdot ||_i$ denotes the Hermitian norm corresponding to H_i.

By the L^2-estimate in Lemma 5.4, we get a section v_i of $K_M^{-\ell}$ such that

$$\bar{\partial} v_i = \bar{\partial} \tilde{\tau}_i$$

and

$$\int_M ||v_i||_i^2 \, \omega_i^n \leq \frac{1}{\ell} \int_M ||\bar{\partial} \tilde{\tau}_i||_i^2 \, \omega_i^n \leq 3 \, \nu \, r^{2n}.$$

Here we have used the fact that $\ell = r^{-2}$.

Put $\sigma_i = \tilde{\tau}_i - v_i$, it is a holomorphic section of $K_M^{-\ell}$. Using (5.14), the L^2-estimate on v_i and the definition of $\tilde{\tau}_i$, we can easily show

$$\int_{M_\infty} ||\sigma_i||_i^2 \, \omega_i^n \leq C, \tag{5.18}$$

where C is independent of i. It follows from (5.16) and (5.10) that the C^2-norm of $\bar{\partial} v_i$ on $\tilde{\phi}_i(U(x;\delta_0))$ is bounded from above by $c \, \delta$ for a uniform constant c. By the standard elliptic estimates, we have

$$\sup_{\tilde{\phi}(U(x;2\delta_0) \cap \phi(B_{10}(o,g_x)))} ||v_i||_i^2 \leq C \, (\delta_0 r)^{-2n} \int_M ||v_i||_i^2 \, \omega_i^n \leq C \, \delta_0^{-2n} \, \nu. \tag{5.19}$$

Note that we always use C to denote a uniform constant. For any given δ_0, if δ and ϵ are sufficiently small, then we can make ν so small that

$$8C \, \nu \leq \delta_0^{2n}.$$

Then we can deduce from (5.16), (5.9) and (5.16)

$$||\sigma_i||_i \geq ||F_i(\tau)||_i - ||v_i||_i > \frac{1}{3} \quad \text{on} \quad \tilde{\phi}_i(U(x;\delta_0) \cap \phi(B_{10}(o, g_x))). \tag{5.20}$$

On the other hand, by applying the derivative estimate in Corollary 4.2 to σ_i, we get

$$\sup_M ||\nabla \sigma_i||_i \,\le\, C' \ell^{\frac{n+1}{2}} \left(\int_M ||\sigma_i||_i^2 \,\omega_i^n\right)^{\frac{1}{2}} \,\le\, C' \, r^{-1}. \tag{5.21}$$

By our choice of ϕ, if ϵ is sufficiently small compared to δ_0, for some $u \in \partial B_1(o, g_x)$, we have

$$d(x, \phi(2\delta_0 u)) \,\le\, d(x, \phi(\epsilon u)) \,+\, d(\phi(\epsilon u), \phi(2\delta_0 u)) \,\le\, 10\,\delta_0\, r.$$

If i is sufficiently large, we deduce from (5.20) and (5.21)

$$||\sigma_i||_i(x_i) \,\ge\, \frac{1}{3} \,-\, C'\,\delta_0,$$

it follows that if we choose δ_0 such that $C'\delta_0 < 1/8$, then $||\sigma_i||_i(x_i) > 1/8$. Combining with (5.18), we see that (5.6) holds. therefore, Theorem 1.2, i.e., the partial C^0-estimate for conic Kähler-Einstein metrics, is proved.

As indicated in [Ti10] and verified in [DS14] for smooth Kähler-Einstein metrics (also see [Li12]), by the arguments in the proof of the partial C^0-estimate, we can prove the following regularity for M_∞:

Theorem 5.9. *The Gromov-Hausdorff limit M_∞ is a normal variety embedded in some $\mathbb{C}P^N$ whose singular set is a subvariety $\bar{\mathcal{S}}$ of complex codimension at least 2.*[21] *If $\beta_\infty < 1$, $\mathcal{S}$ is a subvariety consisting a divisor D_∞ and a subvariety $\bar{\mathcal{S}}$ of complex codimension at least 2. If $\beta_\infty = 1$, $\mathcal{S} = \bar{\mathcal{S}}$. Moreover, D_∞ is the limit of D under the Gromov-Hausdorff convergence.*

Proof. For the readers' convenience, we include a proof. Let us recall some well-known facts (cf, [Ti10]). For any i and sufficiently large ℓ, we can choose an orthonormal basis $\{\sigma_{i,\ell}\}$ of $H^0(M, K_M^{-\ell})$ with respect to ω_i and use this to define a Kodaira embedding

$$\psi_{i,\ell} : M \,\mapsto\, \mathbb{C}P^{N_\ell}, \quad \text{where } N_\ell + 1 \,=\, \dim H^0(M, K_M^{-\ell}).$$

By using the L^2-estimate for $\bar\partial$-operator, we can find an exhaustion of $M_\infty \backslash S$ by open subsets $V_1 \subset V_2 \subset \cdots \subset V_\ell \subset \cdots$ such that $\psi_{i,\ell}$ converge to an embedding

$$\psi_{\infty,\ell} : V_\ell \subset M_\infty \,\mapsto\, \mathbb{C}P^{N_\ell}.$$

By the partial C^0-estimate, there is an integer $m > 0$ such that for any $\ell = mk$, $\psi_{i,\ell}$ converge to an extension of $\psi_{\infty,\ell}$ on M_∞ under the convergence of (M, ω_i) to $(M_\infty, \omega_\infty)$. We still denote this extension by

$$\psi_{\infty,\ell} : \, M_\infty \,\mapsto\, \mathbb{C}P^{N_\ell}.$$

[21] The normality was not necessary for proving Theorem 1.1. In all the applications I know, it suffices to have that $\bar{\mathcal{S}}$ is a subvariety of complex codimension 2.

By the estimate in Corollary 4.2, $\psi_{i,\ell}$ are uniformly Lipschtz, so $\psi_{\infty,\ell}$ is a Lipschtz map.

Claim: M_∞ is a variety.

For this, we only need to show that for $k \geq n+1$, $\psi_{\infty,\ell}$ is a homeomorphism from M_∞ onto its image which is also the limit of complex submanifolds $\psi_{i,\ell}(M) \subset \mathbb{C}P^{N_\ell}$.

By the same arguments as those in proving the partial C^0-estimate, we can show: For any $r > 0$, there are $k(r)$ and $s(k)$ such that if $k \geq k(r)$, then for any $x, y \in M$ such that $d_i(x,y) \geq r$, where $d_i(\cdot,\cdot)$ denotes the distance of the metric ω_i, there is a holomorphic section $\varsigma_i \in H^0(M, K_M^{-\ell})$, where $\ell = mk$, satisfying:

$$\int_M ||\varsigma_i||_i^2 \omega_i^n \,=\, 1 \quad \text{and} \quad |||\varsigma_i||_i(x) \,-\, ||\varsigma_i||_i(y)| \,\geq\, s(k). \tag{5.22}$$

The above claim follows from this and the effective finite generation of the anti-canonical ring of M as shown in the thesis of Chi Li [Li12]. [22] For the orthonormal basis $\{\sigma_{i,a}\}_{0\leq a\leq N_m}$ of $H^0(M, K_M^{-m})$ with respect to ω_i, by the partial C^0-estimate and Corollary 4.2, we have

$$c(m) \,\leq\, \sum_{a=0}^{N_m} ||\sigma_{i,a}||_i^2 \,\leq\, c(m)^{-1}, \tag{5.23}$$

where $c(m)$ is a uniform constant independent of i.

Lemma 5.10. *For any $l \geq 1$ and $\varsigma \in H^0(M, K_M^{-(n+1+l)m})$, there are $h_0, \cdots, h_{N_m}$ in $H^0(M, K_M^{-(n+l)m})$ satisfying:*

$$\varsigma \,=\, \sum_{a=0}^{N_m} h_a\, \sigma_{i,a} \quad \text{and} \quad \int_M ||h_a||_i^2\, \omega_i^n \,\leq\, C(m,l) \int_M ||\varsigma||_i^2\, \omega_i^n, \tag{5.24}$$

where $C(m,l)$ is a constant depending only on $c(m)$, l and n.

This is due to Chi Li (see [Li12], Proposition 7). He proved this by using the Skoda-Siu type estimate (see [Siu08], 2.4).

Note that for any $x \in M_\infty$ and $k \geq 1$, we have

$$\psi_{\infty,mk}^{-1}(\psi_{\infty,mk}(x)) \,\subseteq\, \psi_{\infty,m}^{-1}(\psi_{\infty,m}(x)). \tag{5.25}$$

Using this and Lemma 5.10 with $i \to \infty$, we get

$$\psi_{\infty,m(n+1+l)}^{-1}(\psi_{\infty,m(n+1+l)}(x)) \,\supseteq\, \psi_{\infty,m(n+1)}^{-1}(\psi_{\infty,m(n+1)}(x)).$$

It follows from (5.22) that for any $x \neq y \in M_\infty$,

$$\psi_{\infty,m(n+1+l)}(x) \,\neq\, \psi_{\infty,m(n+1+l)}(y)$$

[22] As I advocated in many occasions before (cf. [Ti10]), the partial C^0-estimate corresponds to an effective version of the finite generation of the anti-canonical ring. Chi Li showed precisely in [Li12] how this works.

if l is sufficiently large. Therefore, we can get

$$\psi_{\infty,m(n+1)}(x) \neq \psi_{\infty,m(n+1)}(y).$$

This implies that $\psi_{\infty,m(n+1)}$ is a homeomorphism, so M_∞ is a variety.

There is another way of proving that $\psi_{\infty,mk}$ is a homeomorphism for k sufficiently large. By (5.25), the composition $\psi_{\infty,m}\cdot\psi_{\infty,mk}^{-1}$ is a well-defined map from the variety Y_{mk} onto Y_m, where

$$Y_{mk} = \lim_{i\to\infty}\psi_{i,mk}(M)\subset\mathbb{C}P^{N_{mk}}, \quad Y_m = \lim_{i\to\infty}\psi_{i,m}(M)\subset\mathbb{C}P^{N_m}.$$

Moreover, this map is also the limit of holomorphic maps $\psi_{i,m}\cdot\psi_{i,mk}^{-1}$, so it is a holomorphic map. Since $\psi_{\infty,m}$ restricted to V_m is an embedding for m sufficiently large, we know that $\psi_{\infty,mk}(\psi_{\infty,m}^{-1}(z))$ is either a point or a connected subvariety in the complex limit space Y_{mk}. The second case can be ruled out by using the fact that there is a bounded function u such that

$$\frac{1}{mk}\omega_{FS}|_{Y_{mk}} = \frac{1}{m}(\psi_{\infty,m}\cdot\psi_{\infty,mk}^{-1})^*(\omega_{FS}|_{Y_m}) + \sqrt{-1}\,\partial\bar\partial\, u,$$

where ω_{FS} always denotes the Fubini-Study metric. This again shows that M_∞ is a variety.

By Theorem 4.3, D converges to a divisor D_∞ in M_∞. Clearly, $\psi_{\infty,m(n+1)}(\mathcal{S})$ contains the singular subvariety $\bar{\mathcal{S}}$ and $\psi_{\infty,m(n+1)}(D_\infty)$ is a divisor of the variety $\psi_{\infty,m(n+1)}(M_\infty)$. We will identify M_∞ with $\psi_{\infty,m(n+1)}(M_\infty)$. We claim that $\mathcal{S}$ coincides with $\bar{\mathcal{S}}$ if $\beta_\infty = 1$ and $\bar{\mathcal{S}}\cup D_\infty$ if $\beta_\infty < 1$. This can be seen as follows: By the partial C^0-estimate, we have a continuous function φ, which is smooth outside $\mathcal{S}$, such that

$$\omega_\infty = \frac{1}{\ell}\omega_{FS}|_{M_\infty} + \sqrt{-1}\,\partial\bar\partial\varphi, \quad \omega_{FS}|_{M_\infty} \le C\,\omega_\infty, \quad \text{where } \ell = m(n+1).$$

Furthermore, we have

$$\left(\frac{1}{\ell}\omega_{FS}|_{M_\infty} + \sqrt{-1}\,\partial\bar\partial\,\varphi\right)^n = ||\sigma_\infty||_0^{-2(1-\beta_\infty)}\, e^{-\mu_\infty\,\varphi}\,\Omega,$$

where σ_∞ is a defining section of D_∞, $\mu_\infty = 1-(1-\beta_\infty)\lambda$ and Ω is a volume form with curvature $\frac{1}{\ell}\omega_{FS}$ and corresponding to a Hermitian metric $||\cdot||_0$ on $K_{M_\infty}^{-1}$.

Near any x outside $\bar{\mathcal{S}}$ if $\beta_\infty = 1$ or $\bar{\mathcal{S}}\cup D_\infty$ if $\beta_\infty < 1$, the right side of above equation is smooth and consequently, ω_∞ is equivalent to ω_{FS} near x. Hence, the regularity theory for complex Monge-Ampere equations on high order derivatives implies that φ is smooth near x and x is outside $\mathcal{S}$. Our claim is proved. Note that $\bar{\mathcal{S}}$ (resp. $\bar{\mathcal{S}}\backslash D_\infty$) is of complex codimension at least 2 if $\beta_\infty = 1$ (resp. $\beta_\infty < 1$).

Next we prove that M_∞ is normal. First we claim that M_∞ is locally connected. This implies that the singularity of M_∞ is of complex codimension at least 2. If $\beta_\infty = 1$, it is trivially true since the singular set of M_∞ is of complex codimension at least 2. So we may assume $\beta_\infty < 1$. There are several approaches. One can use a local version of the Cheeger-Gromoll splitting theorem (cf. [An90]). One can also generalize the arguments I had in [Ti90] or use the Cheeger-Colding theory.

We have shown that the singular set $\mathcal{S}$ of $(M_\infty, \omega_\infty)$ is a subvariety made of the divisor D_∞ possibly plus a subvariety $\bar{\mathcal{S}}$ which is of complex codimension at least 2 outside D_∞. Therefore, if the claim is false, then $M_\infty \backslash D_\infty$ is not locally connected near a point, say x, in D_∞ such that a tangent cone $\mathcal{C}_x$ of M_∞ at x is of the form $\mathbb{C}^{n-1} \times \mathcal{C}'_x$, where $\mathcal{C}'_x$ is a 2-dimensional flat cone of angle $2\pi\bar{\beta}$. However, $\mathcal{C}_x \backslash \mathcal{S}_x$ is connected, so $M_\infty \backslash D_\infty$ is connected near x, a contradiction. Therefore, M_∞ must be locally connected.

Note that the claim can be also deduced from a result of Colding-Naber who proved the convexity of $M_\infty \backslash \mathcal{S}$.

To conclude that M_∞ is normal, we may assume that $M_\infty \subset \mathbb{C}P^N$ and prove that the affine variety $V = M_\infty \backslash H$ is normal for any hyperplane $H \subset \mathbb{C}P^N$. By the general theory in algebraic geometry, we have a normalization $\pi : U \mapsto V$, moreover, U is also an affine variety in some $\mathbb{C}^m$ and π is a finite morphism which is an isomorphism on $\pi^{-1}(V \backslash \mathcal{S})$. Any coordinate function z_i of $\mathbb{C}^m$ $(i = 1, \cdots, m)$ restricts to a holomorphic function f_i on $V \backslash \mathcal{S}$. Since $\mathcal{S}$ is of complex codimension 2, we can show that f_i is bounded. By using this and the formula for $\Delta |f_i|^2$, we can deduce that $|\partial f_i|^2$ is locally integrable. Next, as we did in deriving the partial C^0-estimate, we can show that $|df_i|$ is bounded on any compact subsets. Hence, all f_i $(i = 1, \cdots, m)$ extend to Lipschtz functions on V. This implies that $V = U$, so V, and consequently, M_∞, is normal.

□

Of course, one can further analyze the finer asymptotic structure of ω_∞ along D_∞. For instance, we can show that ω_∞ is a conic Kähler-Einstein metric with cone angle $2\pi\bar{\beta}$ along D_∞ in a weaker sense[23]. It is an interesting problem to examine the precise behavior of ω_∞ along D_∞.

6 Proving Theorem 1.1

In this section, we complete the proof of Theorem 1.1, i.e., if a Fano manifolds M is K-stable, then it admits a Kähler-Einstein metric. As I pointed out in describing my program on the existence of Kähler-Einstein metrics, the method of deriving Theorem 1.1 from the partial C^0-estimate in the context of the Aubin continuity method had been known to me for a long time (cf. [Ti10]). Here, we adapt the argument to the context of the Donaldson-Li-Sun continuity method.

As mentioned in the introduction, the key for proving Theorem 1.1 is to establish the C^0-estimate for the solutions of the complex Monge-Ampere equations for $\beta > 1 - \lambda^{-1}$:

$$(\omega_\beta + \sqrt{-1}\partial\bar{\partial}\varphi)^n = e^{h_\beta - \mu\varphi}\omega_\beta^n, \qquad (6.1)$$

[23] The angle $2\pi\bar{\beta}$ may be different on different connected components of D_∞.

where ω_β is a suitable family of conic Kähler metrics with $[\omega_\beta] = 2\pi c_1(M)$ and cone angle $2\pi\beta$ along D and h_β is determined by

$$\mathrm{Ric}(\omega_\beta) \,=\, \mu\,\omega \,+\, 2\pi(1-\beta)\,[D] \,+\, \sqrt{-1}\,\partial\bar\partial h_\beta \quad \text{and} \quad \int_M (e^{h_\beta}-1)\,\omega_\beta^n \,=\, 0.$$

Let E be the set of $\beta \in (1-\lambda^{-1}, 1]$ such that (6.1) has a solution φ_β. By the discussion in the introduction, we know that E is a non-empty and open interval $E = (1-\lambda^{-1}, \bar\beta)$ for some $\bar\beta \le 1$ or $(1-\lambda^{-1}, 1]$. Actually, such a solution φ_β is unique, so $\{\varphi_\beta\}$ is a continuous family on M and smooth outside D. [24] If we can prove that E is closed, then $E = (1-\lambda^{-1}, 1]$ and the proof of Theorem 1.1 is completed. We will use the K-stability to derive a contradiction if E is not closed.

Now assume $E = (1-\lambda^{-1}, \bar\beta)$ for some $\bar\beta \le 1$. We claim: *If $\bar\beta$ is not in E, $||\varphi_\beta||_{C^0}$ diverge to ∞ as β tends to $\bar\beta$.* If $||\varphi_\beta||_{C^0}$ are uniformly bounded, then we can apply the results in [JMR11] to get a uniform $C^{2,\gamma}$-estimate for φ_β for some $\gamma > 0$. This was done in [JMR11] as follows: Jeffres, Mazzeo and Rubinstein first used the Chern-Lu inequality and the Maximum Principle to bound $\Delta'\varphi_\beta$ uniformly, where Δ' denotes the Laplacian with respect to the conic Kähler-Einstein metric $\tilde\omega_\beta = \omega_\beta + \sqrt{-1}\,\partial\bar\partial\,\varphi_\beta$. This implies that $\tilde\omega_\beta$ is uniformly equivalent to ω_β and (6.1) becomes uniformly elliptic. Then they adapted known techniques for Monge-Ampere equation to conic setting and derived a uniform $C^{2,\gamma}$-estimate for φ_β.[25] Now, by taking a subsequence if needed, φ_β converge to a $C^{2,\gamma}$ function $\varphi_{\bar\beta}$ which satisfies (6.1) with $\beta = \bar\beta$. Then the known regularity theory for conic complex Monge-Ampere equations (see [JMR11]) implies that $\varphi_{\bar\beta}$ is a solution of (6.1) and consequently, $\bar\beta \in E$. This is a contradiction, so our claim is verified.

Our first proof is to use the CM-stability. For simplicity, we first assume that there are no nonzero holomorphic fields on M. Let us recall the CM-stability (cf. [Ti97]). It can be defined in terms of Mabuchi's K-energy:

$$\mathbf{M}_{\omega_0}(\varphi) \,=\, -\frac{1}{V}\int_0^1\int_M \varphi\,(\mathrm{Ric}(\omega_{t\varphi}) - \mu\,\omega_{t\varphi}) \wedge \omega_{t\varphi}^{n-1} \wedge dt. \tag{6.2}$$

Given an embedding $M \subset \mathbb{C}P^N$ by $K_M^{-\ell}$, we have an induced function on $\mathbf{G} = \mathbf{SL}(N+1,\mathbb{C})$ which acts on $\mathbb{C}P^N$:

$$\mathbf{F}(\sigma) \,=\, \mathbf{M}_{\omega_0}(\psi_\sigma), \tag{6.3}$$

where ψ_σ is defined by

$$\frac{1}{\ell}\,\sigma^*\omega_{FS} \,=\, \omega_0 \,+\, \sqrt{-1}\,\partial\bar\partial\psi_\sigma. \tag{6.4}$$

---

[24] In fact, one can prove this continuity and smoothness directly by using the Inverse Function Theorem as one argued for the openness of E.

[25] In [JMR11], they used the Krylov-Evans method in a conic setting for deriving $C^{2,\gamma}$-estimate. However, more arguments are needed in order for them to adapt the Krylov-Evans method to the conic case. In February 14 of 2014, they added an appendix which contains a proof of the required $C^{2,\gamma}$-estimate by using the method from my PKU Master degree thesis. In early 2013, Chen-Donaldson-Sun gave a $C^{2,\gamma}$-estimate for the special case of conic Kähler-Einstein metrics.

Note that $\mathbf{F}(\sigma)$ is well-defined since ψ_σ is unique modulo addition of constants. Similarly, we can define $\mathbf{J}$ on $\mathbf{G}$ by

$$\mathbf{J}(\sigma) = \mathbf{J}_{\omega_0}(\psi_\sigma). \tag{6.5}$$

Definition 6.1. *We call M CM-stable with respect to $K_M^{-\ell}$ if $\mathbf{F}$ is proper, i.e., for any sequence $\sigma_i \in \mathbf{G}$,*

$$\mathbf{F}(\sigma_i) \to \infty \text{ whenever } \mathbf{J}_{\omega_0}(\psi_{\sigma_i}) \to \infty. \tag{6.6}$$

We call M CM-semistable with respect to $K_M^{-\ell}$ if $\mathbf{F}$ is bounded from below. We say M CM-stable (resp. CM-semistable) if it is CM-stable (resp. CM-semistable) with respect to $K^{-\ell}$ for all sufficiently large ℓ.

Remark 6.2. *In [Ti97], the CM-stability is defined in terms of the orbit of a lifting of M in certain determinant line bundle, referred as the CM-polarization. It is proved there that such an algebraic formulation is equivalent to the one in Definition 6.1 (cf. [Ti97], Theorem 8.10).*

The following is a conic version of what I knew for the Aubin's continuity method (cf. [Ti10]).

Theorem 6.3. *If M is a Fano manifold which is CM-stable, then M admits a Kähler-Einstein metric.*

Proof. By the above discussions, if M does not admit any Kähler-Einstein metric, then there is a sequence β_i with $\lim \beta_i = \bar\beta \le 1$ such that the C^0-norms of $\varphi_i = \varphi_{\beta_i}$ diverge to ∞. By the partial C^0-estimate we established in last section, we can have an embedding $M \subset \mathbb{C}P^N$ through a basis of $H^0(M, K_M^{-\ell})$ for some $\ell > 0$ and $\sigma_i \in \mathbf{G}$ such that

$$\psi_i = \psi_{\sigma_i}, \quad ||\psi_i - \varphi_i||_{C^0} \le C. \tag{6.7}$$

Note that C always denotes a uniform constant. We claim:

$$\mathbf{M}_{\omega_0}(\varphi_i) \ge \mathbf{F}(\sigma_i) - C. \tag{6.8}$$

Let us prove this claim. It was shown in [Ti00] (also [LS14], Proposition 3.5) that

$$\mathbf{M}_{\omega_0}(\varphi) = \frac{1}{V}\int_M \log\left(\frac{\omega_\varphi^n}{\omega_0^n}\right)\omega_\varphi^n + (\mathbf{I}_{\omega_0}(\varphi) - \mathbf{J}_{\omega_0}(\varphi)) + \frac{1}{V}\int_M h_0\,(\omega_0^n - \omega_\varphi^n),$$

where h_0 is defined at the beginning of Section 2 and

$$\mathbf{I}_{\omega_0}(\varphi) = \frac{1}{V}\int_M \varphi\,(\omega_0^n - \omega_\varphi^n).$$

It follows from the above and (6.7) that $\mathbf{M}_{\omega_0}(\varphi_i)$ is bounded from below by

$$\begin{aligned} &\mathbf{M}_{\omega_0}(\psi_i) + \frac{1}{V}\int_M \log\left(\frac{\omega_{\psi_i}^n}{\omega_0^n}\right)(\omega_{\varphi_i}^n - \omega_{\psi_i}^n) - C \\ \ge\ &\mathbf{F}(\sigma_i) + \frac{1}{V}\int_M (\varphi_i - \psi_i)\,(\mathrm{Ric}(\omega_0) - \mathrm{Ric}(\omega_{\psi_i})) \wedge \sum_{a=1}^{n-1}\omega_{\varphi_i}^a \wedge \omega_{\psi_i}^{n-a} - C \end{aligned}$$

Then (6.8) follows from this and the fact that $\mathrm{Ric}(\omega_{\psi_i})$ is bounded from above.

Next we recall the twisted K-energy:

$$\mathbf{M}_{\omega_0,\mu}(\varphi) \,=\, \mathbf{M}_{\omega_0}(\varphi) + (1-\mu)(\mathbf{I}_{\omega_0}(\varphi) - \mathbf{J}_{\omega_0}(\varphi)) + \frac{1-\beta}{V}\int_M \log ||S||_0^2\,(\omega_\varphi^n - \omega_0^n).$$

Claim: $\mathbf{M}_{\omega_0,\mu_i}(\varphi_i)$ are uniformly bounded from above. This follows from a known relation between $\mathbf{M}_{\omega_0,\mu}$ and $\mathbf{F}_{\omega_0,\mu}$ (see [LS14], Proposition 2.10, (2)) which generalizes a formula of Ding and myself:

$$\mathbf{M}_{\omega_0,\mu_i}(\varphi_i) \,=\, \mu_i\,\mathbf{F}_{\omega_0,\mu_i}(\varphi_i) \,+\, \frac{1}{V}\int_M (h_0 - (1-\beta_i)\log ||S||_0^2 + a_{\beta_i})\,\omega_0^n,$$

where $\mu_i = 1-(1-\beta_i)\,\lambda$ and a_{β_i} is determined in (2.1). Here we used the fact that ω_{φ_i} is a conic Kähler-Einstein with cone angle $2\pi\beta_i$. Using Lemma 2.4 with $t=\mu_i$ and letting $\delta \to 0$, we get

$$\mathbf{M}_{\omega_0,\mu_i}(\varphi_i) \,\le\, C. \tag{6.9}$$

Since M is CM-stable with respect to $K_M^{-\ell}$, it follows from (6.8) and (6.9) that ψ_i, and consequently, φ_i, are uniformly bounded. This is a contradiction, so our theorem is proved. □

Next we introduce the K-stability. I will use the original one from [Ti97]. First we recall the definition of the Futaki invariant [Fu83]: Let M_0 be any Fano manifold and ω be a Kähler metric with $c_1(M)$ as its Kähler class, for any holomorphic vector field X on M_0, Futaki defined

$$f_{M_0}(X) \,=\, -\,n\int_M \theta_X\,(\mathrm{Ric}(\omega)-\omega)\wedge\omega^{n-1}, \tag{6.10}$$

where $i_X\omega \,=\, \sqrt{-1}\,\bar\partial\theta_X$. Futaki proved in [Fu83] that $f_M(X)$ is independent of the choice of ω, so it is a holomorphic invariant. In [DT92], the Futaki invariant was extended to normal Fano varieties: Assume $M \mapsto \mathbb{C}P^N$ through a basis of $H^0(M, K_M^{-\ell})$ for a sufficiently large ℓ. For any algebraic subgroup $\mathbf{G}_0 \,=\, \{\sigma(t)\}_{t\in\mathbb{C}^*}$ of $\mathbf{G} = \mathbf{SL}(N+1,\mathbb{C})$, there is a unique limiting cycle

$$M_0 \,=\, \lim_{t\to 0}\sigma(t)(M) \subset \mathbb{C}P^N.$$

Let X be the holomorphic vector field whose real part generates the action by $\sigma(e^{-s})$. By [DT92], if M_0 is normal, we can still use (6.10) to define a generalized Futaki invariant $f_{M_0}(X)$. In fact, we only need that M_0 is irreducible in [DT92]. In [Do02], Donaldson gave a formulation of the Futaki invariant $f_{M_0}(X)$ which works for any variety M_0. One can also define $f_{M_0}(X)$ in terms of asymptotic expansion of the K-energy: In his thesis [Li12] (also see [PT06]), Chi Li observed that for any algebraic subgroup $\mathbf{G}_0 = \{\sigma(t)\}_{t\in\mathbb{C}^*}$ of $\mathbf{G}$,

$$\mathbf{F}(\sigma(t)) \,=\, -\,(f_{M_0}(X) \,-\, a(\mathbf{G}_0))\log|t|^2 \,+\, O(1) \;\text{ as }\; t\to 0,$$

where $a(\mathbf{G}_0) \in \mathbb{Q}$ is non-negative and the equality holds if and only if M_0 has no non-reduced components. He also pointed out that (6.11) can be actually derived by using the arguments from [Ti97]. The same arguments can be also used to identify $f_{M_0}(X) - a(\mathbf{G}_0)$ with a Futaki invariant $f_{\tilde{M}_0}(\tilde{X})$ of a $\mathbf{G}_0$-equivariant semi-stable reduction $q : \tilde{\mathcal{X}} \mapsto \mathcal{X}$, where $\mathcal{X} = \{(x,t) \,|\, x \in \sigma(t)(M) \text{ or } M_0\}$, $\tilde{M}_0 = q^{-1}(M_0)$ and $\tilde{X}$ is the field generating the action of $\mathbf{G}_0$ on $\tilde{\mathcal{X}}$. The existence of $\tilde{\mathcal{X}}$ is established in [LX14].[26] In particular, we have

$$\mathbf{F}(\sigma(t)) \,=\, -\operatorname{Re}(f_{\tilde{M}_0}(\tilde{X}))\, \log|t|^2 \,+\, O(1) \quad \text{as} \quad t \to 0. \tag{6.11}$$

One can also prove (6.11) by using the equivariant Riemann-Roch Theorem.

Definition 6.4. *We say that M is K-stable with respect to $K_M^{-\ell}$ if*

$$\operatorname{Re}(f_{M_0}(X)) \,\geq\, 0$$

for any $G_0 \subset \mathbf{SL}(N+1,\mathbb{C})$ with a normal M_0 and the equality holds if and only if M_0 is biholomorphic to M. We say that M is K-stable if it is K-stable for all sufficiently large ℓ.

This was the one given in [Ti97]. There are other formulations of the K-stability by S. Donaldson in [Do02] and S. Paul in [Pa12]. Donaldson's formulation of the K-stability does not require that M_0 is normal. However, by [LX14], Donaldson's formulation is equivalent to Definition 6.4.[27]

It was proved in [Ti97] that *if M is a Fano manifold without non-trivial holomorphic vector fields and admits a Kähler-Einstein metric, then M is K-stable.*

To prove Theorem 1.1, we need to show that if M is a K-stable Fano manifold, then it is CM-stable. In view of (6.11), the K-stability means that $\mathbf{F}$ is proper along any one-parameter algebraic subgroup of $\mathbf{G}$. Hence, by Theorem 6.3, our problem is whether or not the properness of $\mathbf{F}$ on $\mathbf{G}$ follows from the properness of $\mathbf{F}$ along any one-parameter algebraic subgroup of $\mathbf{G}$, or equivalently, the problem is whether or not the CM-stability is the same as the K-stability. This is an algebraic problem in nature. We will prove it by using the approach due to S. Paul [28] and results in [Pa12].

As in classical Geometric Invariant Theory, we deduce the CM-stability from the K-stability in two steps. The following lemma provides the first step.[29]

Lemma 6.5. *Let $\mathbf{T}$ be any maximal algebraic torus of $\mathbf{G}$. If the restriction $\mathbf{F}|_{\mathbf{T}}$ is proper in the sense of* (6.6)*, then M is CM-stable.*

[26] In fact, we only need $\tilde{\mathcal{X}}$ has no multiple components in its central fiber, then one can simply take it as the normalization of a base change of $\mathcal{X}$.

[27] Paul's definition also turns out to be the equivalent.

[28] I learned this approach from S. Paul in the late summer of 2012. In [Pa12] and [Pa13], Paul gave detailed arguments for his approach. Here I used some different arguments which I have been familiar for long.

[29] This step was done more algebraically and differently in [Pa12] or [Pa13].

Proof. We prove it by contradiction. Suppose that we have a sequence $\sigma_i \in \mathbf{G}$ such that $\mathbf{F}(\sigma_i)$ stay bounded while $\mathbf{J}(\sigma_i)$ diverge to ∞.

Recall the Cartan decomposition: $\mathbf{G} = \mathbf{K} \cdot \mathbf{T} \cdot \mathbf{K}$, where $\mathbf{K} = U(N+1)$. Write $\sigma_i = k_i' t_i k_i$ for $k_i, k_i' \in \mathbf{K}$ and $t_i \in \mathbf{T}$. Then we have that $\mathbf{F}(t_i k_i) = \mathbf{F}(\sigma_i)$ stay bounded while $\mathbf{J}(t_i k_i) = \mathbf{J}(\sigma_i)$ diverge to ∞.

On the other hand, since each k_i is represented by unitary matrix, we can show easily

$$|\psi_{t_i} - \psi_{t_i k_i}| \,\le\, \log(N+1).$$

Using the fact that both $\mathrm{Ric}(\omega_{\psi_{t_i}})$ and $\mathrm{Ric}(\omega_{\psi_{t_i k_i}})$ are bounded from above and arguing as in the proof of Theorem 6.3, we can have

$$|\mathbf{F}(t_i) \,-\, \mathbf{F}(t_i k_i)| \,\le\, C.$$

It follows that $\mathbf{F}(t_i)$ stay bounded while $\mathbf{J}(t_i)$ diverge to ∞. We get a contradiction. $\square$

Theorem 6.6. *If a Fano manifold M is K-stable, then it is CM-stable.*[30]

Proof. We will fix an embedding $M \subset \mathbb{C}P^N$ of degree d by using a basis of $H^0(M, K_M^{-\ell})$. By Lemma 6.5, we only need to prove that $\mathbf{F}$ is proper on a maximal algebraic torus $\mathbf{T} \subset \mathbf{G} = \mathbf{SL}(N+1, \mathbb{C})$.

First we recall the Chow coordinate and Hyperdiscriminant of M ([Pa12]): Let $G(N-n-1, N)$ the Grassmannian of all $(N-n-1)$-dimensional subspaces in $\mathbb{C}P^N$. We define

$$Z_M \,=\, \{\, P \in G(N-n-1, N) \,|\, P \cap M \,\neq\, \emptyset \,\}. \tag{6.12}$$

Then Z_M is an irreducible divisor of $G(N-n-1, N)$ and determines a non-zero homogeneous polynomial $R_M \in \mathbb{C}[M_{(n+1)\times(N+1)}]$, unique modulo scaling, of degree $(n+1)d$, where $M_{k\times l}$ denotes the space of all $k \times l$ matrices. We call R_M the Chow coordinate or the M-resultant of M.

Next consider the Segre embedding:

$$M \times \mathbb{C}P^{n-1} \subset \mathbb{C}P^N \times \mathbb{C}P^{n-1} \mapsto \mathbb{P}(M^\vee_{n\times(N+1)}),$$

where $M^\vee_{k\times l}$ denotes its dual space of $M_{k\times l}$. Then we define

$$Y_M \,=\, \{\, H \,\subset\, \mathbb{P}(M^\vee_{n\times(N+1)}) \,|\, T_p(M \times \mathbb{C}P^{n-1}) \,\subset\, H \text{ for some } p \,\}. \tag{6.13}$$

Then Y_M is a divisor in $\mathbb{P}(M^\vee_{n\times(N+1)})$ of degree $\bar{d} \,=\, n^2 d$, and consequently, determines a homogeneous polynomial Δ_M in $\mathbb{C}[M_{n\times(N+1)}]$, unique modulo scaling, of degree $\bar{d}$. We call Δ_M the hyperdiscriminant of M.

Set

$$r = (n+1)\, d\bar{d}, \quad \mathbf{V} \,=\, C_r[M_{(n+1)\times(N+1)}], \quad \mathbf{W} \,=\, C_r[M_{n\times(N+1)}],$$

[30] This theorem is actually true for any polarized manifold and due to S. Paul. Its proof was also given in detail in [Pa13] and [Ti13].

where $C_r[\mathbb{C}^k]$ denotes the space of homogeneous polynomials of degree r on $\mathbb{C}^k$. Following [Pa12], we associate M with the pair $(R(M), \Delta(M))$ in $\mathbf{V}\times\mathbf{W}$, where $R(M) = R_M^{\bar{d}}$ and $\Delta(M) = \Delta_M^{(n+1)d}$.

Fix norms on $\mathbf{V}$ and $\mathbf{W}$, noth denoted by $||\cdot||$ for simplicity, we set

$$p_{v,w} \;=\; \log||w|| \;-\; \log||v||. \tag{6.14}$$

The following was first observed by S. Paul.

Lemma 6.7. *Let $(\sigma, B) \mapsto \sigma(B) : \mathbf{G}\times\mathbf{gl} \mapsto \mathbf{gl}$ be the natural representation by left multiplication, where $\mathbf{gl}$ denotes the space of all $(N+1)\times(N+1)$ matrices. Then we have*

$$|\,\mathbf{J}(\sigma) \;-\; p_{R(M),I^r}(\sigma)\,| \;\leq\; C, \tag{6.15}$$

where I is the identity in $\mathbf{gl}$ and $I^r \in \mathbf{U} = \mathbf{gl}^{\otimes r}$.

Proof. It is known (cf. [Pa04])

$$(n+1)\,\mathbf{J}(\sigma) \;=\; (n+1)\int_M \psi_\sigma\,\omega_0^n \;-\; \log||\sigma(R_M)||^2.$$

This is equivalent to

$$(n+1)\,\bar{d}\,\mathbf{J}(\sigma) \;=\; r\int_M \psi_\sigma\,\frac{\omega_0^n}{d} \;-\; \log||\sigma(R(M))||^2. \tag{6.16}$$

If we write $\sigma \in \mathbf{SL}(N+1,\mathbb{C})$ as a $(N+1)\times(N+1)$-matrix (ϑ_{ij}) with determinant one, then the Hilbert-Schmidt norm of σ is given by

$$||\sigma||^2 \;=\; \sum_{i,j=0}^{N} |\vartheta|^2.$$

Clearly, we have

$$\psi_\sigma \;=\; \log\left(\sum_{i=0}^{N} ||\sum_{j=0}^{N}\vartheta_{ij}S_j||^2\right),$$

where $\{S_j\}_{0\leq j\leq N}$ is an orthonormal basis. By direct computations, we can easily show

$$\left|\log||\sigma||^2 \;-\; \int_M \log\left(\sum_{i=0}^{N}||\sum_{j=0}^{N}\vartheta_{ij}S_j||^2\right)\frac{\omega_0^n}{d}\right| \;\leq\; C.$$

Combining the above two with (6.16), we get (6.15). $\square$

Lemma 6.8. *Let $\mathbf{V}$, $\mathbf{W}$ and $\mathbf{U}$ be as above. If $\mathbf{F}$ is not proper on $\mathbf{T}$ (resp. $\mathbf{G}_0$), then the orbit of $[R(M),\Delta(M)]\times[R(M),I^r]$ under $\mathbf{T}$ (resp. $\mathbf{G}_0$) has a limit point in $(P(\mathbf{V})\times\mathbf{W})\times(\{0\}\times P(\mathbf{U}))$ which is an open subvariety of $P(\mathbf{V}\oplus\mathbf{W})\times P(\mathbf{V}\oplus\mathbf{U})$, where $\mathbf{G}_0$ is an one-parameter algebraic subgroup.*

Proof. First we note that $(P(\mathbf{V})\times\mathbf{W})\times(\{0\}\times P(\mathbf{U}))$ is $\mathbf{T}$-invariant. It follows from [Pa12] that for all $\sigma\in\mathbf{G}$, we have

$$|\,\mathbf{F}(\sigma)\,-\,a_n\,p_{R(M),\Delta(M)}(\sigma)\,|\,\le\,C, \tag{6.17}$$

where $a_n>0$ and C are uniform constants.

By Lemma 6.7 and (6.17), we see that if $\mathbf{F}$ is not proper on $\mathbf{T}$ (resp. $\mathbf{G}_0$), then there are $\sigma_i\in\mathbf{T}$ (resp. $\mathbf{G}_0$) such that $p_{R(M),\Delta(M)}(\sigma_i)$ stay bounded while $p_{R(M),I^r}(\sigma_i)$ goes to ∞. In [Pa12], S. Paul showed

$$p_{R(M),\Delta(M)}(\sigma)\,=\,\log\tan^2 d(\sigma([R(M),\Delta(M)]),\sigma([R(M),0]))$$

and

$$p_{R(M),I^r}(\sigma)\,=\,\log\tan^2 d(\sigma([R(M),I^r]),\sigma([R(M),0])),$$

where $d(\cdot,\cdot)$ denotes the distance in $P(\mathbf{V}\oplus\mathbf{W})$ with respect to the Fubini-Study metric. Therefore, the limits of $\sigma_i([R(M),I^r])$ lie in $\{0\}\times P(\mathbf{U})$ while limits of $\sigma_i([R(M),\Delta(M)])$ stay in $P(\mathbf{V})\times\mathbf{W}$. The lemma is proved. □

Now we deduce Theorem 6.6 from Lemma 6.8. If M is not CM-stable, then there are $v\in\mathbf{V},w\in\mathbf{W},u\in\mathbf{U}$ such that $u,v\neq 0$ and $\bar{y}=[v,w]\times[0,u]$ is in the closure of the $\mathbf{T}$-orbit of $x=[R(M),\Delta(M)]\times[R(M),I^r]$. Choose $\mathbf{T}$-invariant hyperplanes $\mathbf{V}_0\subset\mathbf{V}$ and $\mathbf{U}_0\subset\mathbf{U}$, which can be naturally identified with $P(\mathbf{V})\backslash P(\mathbf{V}_0)$ and $P(\mathbf{U})\backslash P(\mathbf{U}_0)$, such that $x\in\mathbf{E}=\mathbf{V}_0\times\mathbf{W}\times\mathbf{V}\times\mathbf{U}_0$ and $y\in\mathbf{E}_0=\mathbf{V}_0\times\mathbf{W}\times\{0\}\times\mathbf{U}_0$. Clearly, the orbit $\mathbf{T}\cdot y$ lies in the closed subspace $\mathbf{E}_0$ of $\mathbf{E}$. By taking an orbit in the closure of $\mathbf{T}\cdot y$ if necessary, we may assume that $\mathbf{T}\cdot y$ is closed in $\mathbf{E}_0$. Then, by a well-known result of Richardson (cf. [Pa12] and also [Ti13]), there is an one-parameter algebraic subgroup $\mathbf{G}_0$ such that the closure of $\mathbf{G}_0\cdot x$ contains a point in $\mathbf{E}_0$ which is a subset of $(P(\mathbf{V})\times\mathbf{W})\times(\{0\}\times P(\mathbf{U}))$. By Lemma 6.8, this contradicts to the K-stability of M. Thus, the proof of Theorem 6.6 is completed. □

Theorem 1.1 follows from Theorem 6.3 and Theorem 6.6.

If M has non-zero holomorphic vector fields, instead of proving (6.6), we prove that for any sequence $\sigma_i\in\mathbf{G}$,

$$\mathbf{F}(\sigma_i)\to\infty \text{ whenever } \inf_{\tau\in Aut_0(M)}\mathbf{J}(\sigma_i\tau)\to\infty, \tag{6.18}$$

where $Aut_0(M)$ denotes the identity component of the automorphism group of M. One can modify the above arguments to prove (6.18) when M is K-stable. So we can still prove Theorem 1.1 in general cases by using the CM-stability.

There is another way of completing the proof of Theorem 1.1.[31] If M does not admit any Kähler-Einstein metric, we have a sequence $\beta_i\in E$ which converge to $\bar{\beta}\notin E$ which is not in E. Then, by taking a subsequence if necessary,

[31] This proof was outlined in the first version of this paper and is actually simpler.

we may assume that (M, D, ω_i) converge to $(M_\infty, D_\infty, \omega_\infty)$. By using the partial C^0-estimate established in last section, we may further have that (1) M is embedded in $\mathbb{C}P^N$ through an orthonormal basis of $H^0(M, K_M^{-\ell})$ given by ω_i; (2) $M_\infty \subset \mathbb{C}P^N$ is a normal subvariety with a divisor D_∞; (3) There are $\sigma_i \in \mathbf{G} = \mathbf{SL}(N+1, \mathbb{C})$ such that $(\sigma_i(M), \sigma_i(D))$ converge to (M_∞, D_∞). It follows that the stabilizer $\mathbf{G}_\infty$ of (M_∞, D_∞) in $\mathbf{G}$ contains a non-trivial holomorphic subgroup.

Lemma 6.9. *The Lie algebra η_∞ of $\mathbf{G}_\infty$ is reductive.*

Proof. The proof has two steps: In the first step, we prove that any holomorphic field in η_∞ is a complexification of a Killing field on M_∞. [32] In the second and easy step, we show that any Killing field can be extended to be the imaginary part of a holomorphic field on the ambient projective space.

First we state a technical result which I knew for long. As usual, $\bar{\mathcal{S}}$ denotes the singular set of M_∞. Since $\bar{\mathcal{S}} \cup D_\infty \subset M_\infty \subset \mathbb{C}P^N$ is a subvariety, there is a holomorphic section $\tilde{\tau} \in H^0(\mathbb{C}P^N, \mathcal{O}(k))$ which vanishes on $\bar{\mathcal{S}} \cup D_\infty$ for some k. Note that $\tilde{\tau}$ is actually given by a homogeneous polynomial of degree k on $\mathbb{C}^{N+1}$. Also we have $K_{M_\infty}^{-\ell'} = \mathcal{O}(k)|_{M_\infty}$ for $\ell' = k\ell$, so we get a holomorphic section τ_∞ in $H^0(M_\infty, K_{M_\infty}^{-\ell'})$ whose zero set contains $\bar{\mathcal{S}} \cup D_\infty$, in particular, $M_\infty \backslash \tau_\infty^{-1}(0)$ is contained in the regular part of $(M_\infty, \omega_\infty)$. Choose a cut-off function $\tilde{\eta} : \mathbb{R} \mapsto \mathbb{R}$ satisfying: $\tilde{\eta}(t) = 1$ for $t \geq 2$, $\tilde{\eta}(t) = 0$ for $t \leq 1$, $|\tilde{\eta}'(t)| \leq 1$ and $|\tilde{\eta}''(t)| \leq 4$. For any $\epsilon > 0$, we define

$$\gamma_\epsilon(x) \,=\, \tilde{\eta}(\epsilon \, \log(-\log ||\tau_\infty||_0^2(x)),$$

where $||\cdot||_0$ denotes the Hermitian norm with curvature ω_{FS} restricted to M_∞.

Lemma 6.10. *Let $\psi_1, \cdots, \psi_k$ $(k \leq n-1)$ be bounded functions which are smooth and satisfy: $\omega_i = \frac{1}{\ell}\,\omega_{FS} + \sqrt{-1}\,\partial\bar\partial\,\psi_i \geq 0$ outside $\bar{\mathcal{S}} \cup D_\infty$. Then*

$$\lim_{\epsilon \to 0} \int_{M_\infty} \sqrt{-1}\,\partial\gamma_\epsilon \wedge \bar\partial\gamma_\epsilon \wedge \omega_1 \wedge \cdots \wedge \omega_k \wedge \left(\frac{1}{\ell}\,\omega_{FS}\right)^{n-1-k} \,=\, 0. \qquad (6.19)$$

This lemma should be also known to experts in studying the theory on plurisubharmonic functions since its proof uses the standard arguments in studying (1,1)-currents with locally bounded potentials. For the readers' convenience, we will give a proof at the end of this section.

Let X be a holomorphic vector field on $\mathbb{C}P^N$ which is tangent to M_∞. We will show that there is a bounded function θ_∞ such that $i_X\omega_\infty \,=\, \sqrt{-1}\,\bar\partial\theta_\infty$ on $M_\infty \backslash \bar{\mathcal{S}} \cup D_\infty$. Let ϕ_t be an one-parameter subgroup of automorphisms generated by Y which is either the real or imaginary part of X, then we have

$$\phi_t^*\omega_\infty \,=\, \frac{1}{\ell}\,\omega_{FS} \,+\, \sqrt{-1}\,\partial\bar\partial\,\psi_t.$$

[32] In [BB12], based on arguments from [Bo11], Berman-Boucksom-Essydieux-Guedj-Zeriahi proved a strong uniqueness theorem: Kähler-Einstein metrics on (possibly) singular Fano varieties are unique modulo an automorphism group obtained by complexifying an isometry group. The step one, and consequently, Lemma 6.9, follows directly from this uniqueness result. This observation was first pointed out in writing by Chen-Donaldson-Sun [CDS15, I, II, III].

Since ω_∞ is a weakly Kähler-Einstein metric, we may choose ψ_t such that

$$\phi_t^*\omega_\infty^n \,=\, (\omega_\infty \,+\, \sqrt{-1}\,\partial\bar\partial\,\xi_t)^n \,=\, e^{-\bar\mu\,\xi_t}\,\omega_\infty^n \quad \text{on } \; M_\infty\backslash\bar{\mathcal{S}}\cup D_\infty, \qquad (6.20)$$

where $\xi_t = \psi_t - \psi_0$ and $\bar\mu = 1 - (1-\bar\beta)\,\lambda$. Each ψ_t, or equivalently ξ_t, is a bounded, actually continuous, function. The continuity follows from the partial C^0-estimate. To see this, we note

$$\omega_\infty = \frac{1}{\ell}\,\omega_{FS} + \sqrt{-1}\partial\bar\partial\psi_0.$$

Then we have

$$\phi_t^*\omega_\infty = \frac{1}{\ell}\,\phi_t^*\omega_{FS} + \sqrt{-1}\partial\bar\partial\psi_0\circ\phi_t = \frac{1}{\ell}\,\omega_{FS} + \sqrt{-1}\partial\bar\partial\psi_t.$$

It implies

$$\xi_t \,=\, \psi_t - \psi_0 = \psi_0\circ\phi_t - \psi_0 + \zeta_t,$$

where ζ_t is a smooth function on $\mathbb{C}P^N$ and satisfies

$$\phi_t^*\omega_{FS} \,=\, \omega_{FS} + \ell\sqrt{-1}\,\partial\bar\partial\zeta_t, \qquad \zeta_0 = 0.$$

The partial C^0-estimate implies that a subsequence of $\{\log\rho_{i,\ell}\}$ converges to $\ell\,\psi_0 + c$ for some constant c as (M,ω_i) converge to (M_∞,ω_∞), where $\rho_{i,\ell}$ are defined in (5.3) as the sum of square norms of sections in an orthonormal basis of $H^0(M,K_M^{-\ell})$ with respect to ω_i, e.g., the one in above (1). By the gradient estimate in Corollary 4.2, sections in such a basis are uniformly continuous, so $\rho_{i,\ell}$ are uniformly continuous for any fixed ℓ. Since $\rho_{i,\ell}$ are uniformly bounded by a positive constant, it follows that ψ_0 is continuous, so does each ξ_t. We also see that $|\xi_t| \le 1/2$ if t is sufficiently small.

It follows from (6.20) that

$$-\sqrt{-1}\,\partial\bar\partial\,\xi_t\wedge\left(\sum_{i=0}^{n-1}\omega_\infty^i\wedge(\phi_t^*\omega_\infty)^{n-i-1}\right) \,=\, (1-e^{-\bar\mu\,\xi_t})\,\omega_\infty^n. \qquad (6.21)$$

We multiply this by $\gamma_\epsilon\xi_t$ and integrate by parts, then by using Lemma 6.10 as ϵ goes to 0, we get

$$\int_{M_\infty}\sqrt{-1}\,\partial\xi_t\wedge\bar\partial\xi_t\,\left(\sum_{i=0}^{n-1}\omega_\infty^i\wedge(\phi_t^*\omega_\infty)^{n-i-1}\right) \,=\, \int_{M_\infty}\xi_t\,(1-e^{-\bar\mu\,\xi_t})\,\omega_\infty^n.$$

It follows easily

$$\frac{1}{n}\int_{M_\infty}|\nabla\xi_t|^2\,\omega_\infty^n \,\le\, 2\,\bar\mu\int_{M_\infty}|\xi_t|^2\,\omega_\infty^n \qquad (6.22)$$

whenever we choose t so small that $|\xi_t|\le 1/2$. Set

$$\bar\gamma_\delta(x) \,=\, 1 \,-\, \tilde\eta(\delta^{-1}\,||\tau_\infty||_0(x)).$$

Then $\bar{\gamma}_\delta(x)$ is equal to 1 when $||\tau_\infty||(x) \le \delta$ and supports in the subset E_δ where $||\tau_\infty||_0$ is not greater than 2δ. Then we have

$$\int_{M_\infty} |\nabla(\bar{\gamma}_\delta \xi_t)|^2 \,\omega_\infty^n \,\le\, 3n \int_{M_\infty} |\bar{\gamma}_\delta \xi_t|^2 \,\omega_\infty^n \,+\, t^2\, C_\delta, \tag{6.23}$$

where C_δ denotes a constant which depends only on δ. Let me explain why this is true: Recall

$$\xi_t = \psi_t - \psi_0 = \psi_0 \circ \phi_t - \psi_0 + \zeta_t.$$

Since ζ_t is defined on $\mathbb{C}P^N$ by $\phi_t^* \omega_{FS} \,=\, \omega_{FS} + \ell\sqrt{-1}\,\partial\bar{\partial}\zeta_t$ and $\zeta_0 = 0$, we have $|\zeta_t| \,\le\, \sup|\dot{\zeta}_t|\, t \,\le\, C_\zeta\, t$ for some constant C_ζ independent of t, where $\dot{\zeta}_t$ denotes the derivative of ζ_t on t. On the other hand, since ψ_0 is smooth outside $\bar{\mathcal{S}} \cup D_\infty$, we have $|d\psi_0| \le C'_\delta$ outside $E_{\delta/4}$ for some C'_δ which may depend on δ. Using the fact that ϕ_0 is the identity map, for t small, we have

$$|\xi_t| \,\le\, |\psi_0 \circ \phi_t - \psi_0| + |\zeta_t| \,\le\, (C'_\delta \sup_{M_\infty \backslash E_{\delta/4}} |\dot{\phi}_t| + C_\zeta)\, t \ \text{ on } M_\infty \backslash E_{\delta/2},$$

where $\dot{\phi}_t$ denotes the t-derivative of ϕ_t. Hence, we have

$$\int_{M_\infty \backslash E_{\delta/2}} |\xi_t|^2 \,\omega_\infty^n \,\le\, C''_\delta\, t^2. \tag{6.24}$$

Using the Cauchy-Schwartz inequality and the fact $\bar{\gamma}_\delta = 1$ on $E_{\delta/2}$, we get

$$\int_{M_\infty} |\nabla(\bar{\gamma}_\delta \xi_t)|^2 \,\omega_\infty^n \,\le\, \frac{3}{2} \int_{M_\infty} |\nabla \xi_t|^2 \,\omega_\infty^n \,+\, 10 \int_{M_\infty} |\nabla \bar{\gamma}_\delta|^2 \,|\xi_t|^2 \,\omega_\infty^n.$$

Then (6.23) follows from this, (6.22) and (6.24).

For any open $E \subset M_\infty$ with nonempty boundary $\partial E \subset M_\infty \backslash \mathcal{S}$, we define

$$\lambda_1(E) \,=\, \inf\left\{ \frac{\int_E |\nabla v|^2 \,\omega_\infty^n}{\int_E |v|^2 \,\omega_\infty^n} \,\Big|\, 0 \ne v \,\in\, C^1(E\backslash\mathcal{S}) \cap L^\infty(E),\, v|_{\partial E} = 0 \right\}.$$

We call it the first eigenvalue of (E, ω_∞) with vanishing boundary condition.

Claim: $\lambda_1(E_\delta) \,\ge\, 4n$ if δ is sufficiently small.

Proof. Our claim is a consequence of Proposition 6 in [Li80] and Theorem 11 in [Cr80] adapted to our situation.

First we claim that $Vol(E_\delta) \to 0$ as $\delta \to 0$, where $Vol(E_\delta)$ is the Hausdorff measure of E_δ associated to the metric structure of $(M_\infty, \omega_\infty)$. Since $\mathcal{S}$ has codimension at least 2 (cf. Theorem 3.1), we have

$$Vol(E_\delta) \,=\, \int_{E_\delta \backslash \mathcal{S}} \omega_\infty^n.$$

Moreover, for any $\bar{\epsilon} > 0$, we can choose a small neighborhood $U_{\bar{\epsilon}}$ of $\mathcal{S}$ such that $Vol(U_{\bar{\epsilon}}) \,\le\, \bar{\epsilon}$. Note that ω_∞ is smooth in an open set containing $M_\infty \backslash U_{\bar{\epsilon}}$, so if

δ is sufficiently small, we have $Vol(E_\delta \backslash U_{\bar{\epsilon}}) \le \bar{\epsilon}$. It follows $Vol(E_\delta) \le 2\,\bar{\epsilon}$, so our claim is proved.

Secondly, we note that it suffices to take v with support away from $\mathcal{S}$ in the definition of $\lambda_1(E_\delta)$. Given any v in defining $\lambda_1(E_\delta)$ with $\int_{E_\delta} |\nabla v|^2 \omega_\infty^n < \infty$, let γ_ϵ be the cut-off functions in Lemma 6.10, then $\gamma_\epsilon\, v$ have supports away from $\mathcal{S}$ and satisfy

$$\int_{E_\delta} |\nabla(\gamma_\epsilon\, v)|^2\, \omega_\infty^n \;=\; \int_{E_\delta} \left(\gamma_\epsilon^2\, |\nabla v|^2 \;+\; 2 < v\, \nabla\gamma_\epsilon, \gamma_\epsilon\, \nabla\; v > \,+ v^2\, |\nabla \gamma_\epsilon|^2 \right) \omega_\infty^n.$$

Since v is bounded, we have

$$\lim_{\epsilon \to 0} \left| \int_{E_\delta} v^2\, |\nabla\gamma_\epsilon|^2\, \omega_\infty^n \right| \;=\; 0.$$

The Cauchy-Schwartz inequality gives

$$\left| \int_{E_\delta} < v\, \nabla\gamma_\epsilon, \gamma_\epsilon\, \nabla\; v > \omega_\infty^n \right| \le \left(\int_{E_\delta} v^2\, |\nabla\gamma_\epsilon|^2\, \omega_\infty^n \right)^{\frac{1}{2}} \left(\int_{E_\delta} \gamma_\epsilon^2\, |\nabla v|^2\, \omega_\infty^n \right)^{\frac{1}{2}}.$$

Therefore, we have

$$\lim_{\epsilon \to 0} \int_{E_\delta} |\nabla(\gamma_\epsilon\, v)|^2\, \omega_\infty^n \;=\; \int_{E_\delta} |\nabla v|^2\, \omega_\infty^n.$$

So we may assume that v supports away from $\mathcal{S}$ in estimating $\lambda_1(E_\delta)$ from below.

By applying Theorem 2.6 to each (M, ω_i), we can find smooth Kähler metrics $\tilde{\omega}_i$ on M with Ricci curvature $\mathrm{Ric}(\tilde{\omega}_i) \ge \mu_i\, \tilde{\omega}_i$, where $\lim \mu_i \;=\; \bar{\mu}$, such that $(M, \tilde{\omega}_i)$ converge to $(M_\infty, \omega_\infty)$ in the Cheeger-Gromov topology. Choose smooth domains $E_\delta^i \subset M$ such that they converge to E_δ as $(M, \tilde{\omega}_i)$ converge to $(M_\infty, \omega_\infty)$. For any $v \in C^1(M_\infty \backslash \mathcal{S})$ with support away from $\mathcal{S}$, we can find diffeomorphisms $\phi_i : M_\infty \backslash \overline{U} \mapsto M \backslash D$, where U is a small neighborhood of $\mathcal{S}$, such that $v = 0$ near $\overline{U} \subset E_\delta$ and $\phi_i^* \tilde{\omega}_i$ converge to ω_∞ on $M_\infty \backslash \bar{U}$ in the smooth topology. For each i, put $v_i = v \circ \phi_i^{-1}$, then by defining $v_i = 0$ on $E_\delta^i \backslash \phi_i(U)$, we get a sequence of smooth functions v_i on E_δ^i converging to v. It follows

$$\lambda_1(E_\delta) \;\ge\; \inf \lim_{i \to \infty} \lambda_1(E_\delta^i),$$

where $\lambda_1(E_\delta^i)$ is the first eigenvalue of $(E_\delta^i, \tilde{\omega}_i)$ with zero boundary condition.

By Proposition 6 in [Li80], we have

$$\lambda_1(E_\delta^i) \;\ge\; \left(\frac{c(n)\, C_{s,i}}{Vol_i(E_\delta^i)} \right)^{1/n},$$

where $c(n)$ is a constant depending only on n, $C_{s,i}$ is the Sobolev constant of $\tilde{\omega}_i$ for functions with compact support in E_δ^i and $Vol_i(E_\delta^i)$ denotes the volume of E_δ^i with respect to the metric $\tilde{\omega}_i$. Since $C_{s,i}$ is equivalent to the isoperimetric constant of $(E^i_\delta, \tilde{\omega}_i)$, Theorem 11 in [Cr80] yields that $C_{s,i}$ can be uniformly bounded from below by lower bound μ_i of Ricci curvature, diameter and the volume of $(M, \tilde{\omega}_i)$. On the other hand, it follows from the result on volume convergence in the Gromov-Hausdorff topology in [CC97] that $Vol_i(E^i_\delta)$ converge to $Vol(E_\delta)$ as i goes to ∞, particularly, $Vol_i(E^i_\delta)$ is small if δ is sufficiently small and i is sufficiently large. Thus $\lambda_1(E_\delta) \geq 4n$ if δ is taken sufficiently small, so our claim is proved.

There is an alternative way of proving our claim without using those $\tilde{\omega}_i$. Let $\epsilon > 0$ be much smaller than δ, it follows from the second observation above that

$$\lambda_1(E_\delta) \,\geq\, \inf \lim_{\epsilon\to 0} \lambda_1(E_\delta \backslash E_\epsilon).$$

Then, by applying Proposition 6 in [Li80] and Theorem 11 in [Cr80] to $E_\delta \backslash E_\epsilon$ and arguing as we did for E^i_δ above, we get $\lambda_1(E_\delta \backslash E_\epsilon) \geq 4n$ if δ is sufficiently small. Thus we give another proof of our claim. □

It follows from (6.23) and the above claim

$$n \int_{M_\infty} |\bar{\gamma}_\delta \xi_t|^2 \, \omega_\infty^n \,\leq\, C_\delta \, t^2.$$

Since $\bar{\gamma}_\delta = 1$ on $E_{\delta/2}$, we deduce from (6.24)

$$\int_{M_\infty} |(1-\bar{\gamma}_\delta)\,\xi_t|^2 \, \omega_\infty^n \,\leq\, \int_{M_\infty \backslash E_{\delta/2}} |\xi_t|^2 \, \omega_\infty^n \,\leq\, C''_\delta \, t^2.$$

It follows from the Cauchy-Schwartz inequality and the above two inequalities

$$\int_{M_\infty} |\xi_t|^2 \, \omega_\infty^n \,\leq\, 2\left(\int_{M_\infty} |\bar{\gamma}_\delta \xi_t|^2 \, \omega_\infty^n + \int_{M_\infty} |(1-\bar{\gamma}_\delta)\xi_t|^2 \, \omega_\infty^n \right) \,\leq\, \bar{C}_\delta \, t^2.$$

Combining this with (6.22) and dividing by t^2, we get

$$\int_{M_\infty} \left(|t^{-1} \nabla \xi_t|^2 \,+\, |t^{-1}\xi_t|^2 \right) \, \omega_\infty^n \,\leq\, 2\,C_\delta. \tag{6.25}$$

First we assume that Y is the real part of X. Since $\xi_t = \psi_0 \circ \phi_t - \psi_0 + \zeta_t$ and ψ_0 is smooth outside $\bar{S} \cup D_\infty$, we see that $t^{-1}\xi_t$ converge pointwisely to u on $M_\infty \backslash \bar{S} \cup D_\infty$ as $t \to 0$. Letting t go to 0, we deduce from (6.25)

$$\int_{M_\infty} |u|^2 \, \omega_\infty^n \,\leq\, C. \tag{6.26}$$

Moreover, by differentiating (6.20) on t, we have

$$\int_{M_\infty} u \, \omega_\infty^n \,=\, 0.$$

For any $q \geq 2$, we multiply (6.21) by $\gamma_\epsilon \xi_t |\xi_t|^{q-2}$ and integrate by parts, then by letting ϵ go to 0, we get

$$\int_{M_\infty} |\nabla |t^{-1}\xi_t|^{\frac{q}{2}}|^2 \,\omega_\infty^n \;\leq\; \frac{12\, n\, \bar{\mu}\, q^2}{q-1} \int_{M_\infty} |t^{-1}\xi_t|^q \,\omega_\infty^n. \tag{6.27}$$

applying the Sobolev inequality to its LHS, we get

$$\left(\int_{M_\infty} |t^{-1}\xi_t|^{\frac{qn}{n-1}} \,\omega_\infty^n \right)^{\frac{n-1}{n}} \;\leq\; C\, q \int_{M_\infty} |t^{-1}\xi_t|^q \,\omega_\infty^n,$$

where C is a constant depending only on the Sobolev constant, n and $\bar{\mu}$. It is shown in (6.25) that $t^{-1}\xi_t$ has uniformly bounded L^2-norm. Therefore, by starting with $q_0 = 2$ and iterating with $q_{i+1} = \frac{q_i n}{n-1}$ for $i \geq 0$, we get that for any $q \geq 2$, there is a uniform constant C_q satisfying:

$$\int_{M_\infty} |t^{-1}\xi_t|^q \,\omega_\infty^n \leq C_q.$$

This implies that $t^{-1}\xi_t$ converge to u in any L^q-norm. To see this, we fix $q \geq 2$. It follows from the Hölder inequality that for any $\delta > 0$,

$$\int_{E_\delta} |t^{-1}\xi_t|^q \,\omega_\infty^n \leq Vol(E_\delta)^{\frac{1}{n}} \left(\int_{M_\infty} |t^{-1}\xi_t|^{\frac{qn}{n-1}} \,\omega_\infty^n \right)^{\frac{n-1}{n}} \leq Vol(E_\delta)^{\frac{1}{n}} (C_{\frac{qn}{n-1}})^{\frac{n-1}{n}}.$$

So for any $\epsilon > 0$, we can choose δ sufficiently small such that

$$\int_{E_\delta} |t^{-1}\xi_t|^q \,\omega_\infty^n \;\leq\; \frac{\epsilon}{3}. \tag{6.28}$$

Since $t^{-1}\xi_t$ converge to u outside D_∞, for t sufficiently small, we have

$$\int_{M \setminus E_\delta} |t^{-1}\xi_t - u|^q \,\omega_\infty^n \;\leq\; \frac{\epsilon}{3}.$$

By letting t go to 0, we deduce from (6.28)

$$\int_{E_\delta} |u|^q \,\omega_\infty^n \;\leq\; \frac{\epsilon}{3}.$$

Putting the above three estimates together and letting ϵ go to 0, we see that $t^{-1}\xi_t$ converge to u in the L^q-norm.

By taking t go to 0, we can now deduce from (6.27)

$$\int_{M_\infty} |\nabla |u|^{\frac{q}{2}}|^2 \,\omega_\infty^n \;\leq\; \frac{12\, n\, \bar{\mu}\, q^2}{q-1} \int_{M_\infty} |u|^q \,\omega_\infty^n.$$

Then, by using (6.26) and the standard Moser iteration, we can easily prove that u is bounded.

Each ψ_t is smooth outside $\bar{\mathcal{S}} \cup D_\infty$ and satisfies:

$$\frac{1}{\ell}\,\omega_{FS} \,+\, \sqrt{-1}\,\partial\bar\partial\psi_t \,=\, \phi_t^*\omega_\infty \,=\, \frac{1}{\ell}\,\phi_t^*\omega_{FS} \,+\, \sqrt{-1}\,\partial\bar\partial\,\phi_t^*\psi_0.$$

It follows that $\psi_t = \phi_t^*\psi_0 + \zeta_t$, where $\phi_t^*\omega_{FS} = \omega_{FS} + \ell\sqrt{-1}\,\partial\bar\partial\,\zeta_t$. Note that ζ_t is a smooth function on the whole $\mathbb{C}P^N$ as well as in t. Thus, we have

$$u \,=\, Y(\psi_0) \,+\, \theta_u, \quad \text{where } \ \theta_u \,=\, \frac{\partial\zeta}{\partial t}\Big|_{t=0}.$$

Similarly, by taking Y to be the imaginary part of X, we can get a bounded function $v = Y(\psi_0) + \theta_v$.

Set $\theta_\infty \,=\, u + \sqrt{-1}\,v$ and $\theta \,=\, \theta_u \,+\, \sqrt{-1}\,\theta_v$, then $\theta_\infty \,=\, X(\psi_0) + \theta$ is a bounded function on M_∞ and $i_X\omega_{FS} \,=\, \ell\sqrt{-1}\,\bar\partial\,\theta$ holds on $\mathbb{C}P^N$. Clearly, we have $i_X\omega_\infty \,=\, \sqrt{-1}\,\bar\partial\,\theta_\infty$. Moreover, we have

$$\int_{M_\infty} |\nabla\theta_\infty|^2\,\omega_\infty^n \,<\, \infty \quad \text{and} \quad \int_{M_\infty} \theta_\infty\,\omega_\infty^n \,=\, 0. \tag{6.29}$$

Next we show that θ_∞ satisfies an eigenfunction equation in a weak sense. Let ϕ_t be as above and ζ be any function on M_∞ which can be extended to be a smooth function in an neighborhood of M_∞ in $\mathbb{C}P^N$. It follows from (6.20) and change of variables that

$$\int_{M_\infty} \zeta\circ\phi_t^{-1}\,\omega_\infty^n \,=\, \int_{M_\infty} \zeta\, e^{-\bar\mu\,\xi_t}\,\omega_\infty^n.$$

This is equivalent to

$$\int_{M_\infty}\left(\int_0^t Y(\zeta)\circ\phi_s\,ds\right)\omega_\infty^n \,=\, \bar\mu\int_{M_\infty}\zeta\,\left(\int_0^t \dot\xi\,ds\,\wedge\,\phi_s^*\omega_\infty^n\right). \tag{6.30}$$

where $\dot\xi$ denotes the t-derivative of ξ_t.

Dividing (6.30) by t and taking Y to be the real or imaginary part of X, as t tends to 0, we deduce

$$\int_{M_\infty} X(\zeta)\,\omega_\infty^n \,=\, \bar\mu\int_{M_\infty}\zeta\,\theta_\infty\,\omega_\infty^n.$$

That is, in the weak sense,

$$-\,\Delta_\infty\theta_\infty \,=\, \bar\mu\,\theta_\infty \quad \text{on} \ \ M_\infty. \tag{6.31}$$

On the other hand, by our assumption on $\bar\beta$ and Theorem 2.6 in Section 2, there are smooth Kähler manifolds $(M,\tilde\omega_i)$ with $\mathrm{Ric}(\tilde\omega_i) \,\ge\, \mu_i\,\tilde\omega_i$ and converging to (M_∞,ω_∞) in the Cheeger-Gromov topology, where $\lim\mu_i = \bar\mu$.

Claim: *Any bounded eigenfunction satisfying* (6.31) *is the limit of eigenfunctions θ_i on M such that $\Delta_i\theta_i \,=\, -\lambda_i\,\theta_i$ with $\lim\lambda_i = \bar\mu$, where Δ_i denotes the Laplacian of $\tilde\omega_i$.*

This is well-known if $(M_\infty, \omega_\infty)$ is smooth since the spectra depend continuously on smooth metrics on a manifold. Clearly, in view of Lemma 6.10, the arguments for smooth metrics apply to our case. Identical arguments were also used in my previous works. For the readers' convenience, we include a proof here by using Lemma 6.10 and standard arguments.

First we need to consider only real eigenfunctions since Δ_i are real operators. Secondly, since $(M, \tilde\omega_i)$ has uniform Sobolev inequality and smooth away from $\mathcal{S}$, we have: For any $\{u_i\}$ with $-\Delta_i\, u_i = \mu_i\, u_i$ and $||u_i||_{L^2} = 1$, by taking a subsequence if necessary, u_i converge to a function u with $-\Delta_\infty\, u = \bar\mu\, u$ and $||u||_{L^2} = 1$. Let $\tilde\Lambda_{\bar\mu}$ be the set of all such u, then it is a subspace of $\Lambda_{\bar\mu}$ which consists of all bounded eigenfunctions with eigenvalue $\bar\mu$. If $\tilde\Lambda_{\bar\mu} \neq \Lambda_{\bar\mu}$, then there is a bounded $u \in \Lambda_{\bar\mu}$ such that

$$\int_{M_\infty} u^2\, \omega_\infty^n \,=\, 1, \quad \int_{M_\infty} u\, u_a\, \omega_\infty^n \,=\, 0, \quad \int_{M_\infty} |\nabla u|^2\, \omega_\infty^n \,=\, \bar\mu,$$

where $\{u_a\}_{1\le a\le k}$ is an orthonormal basis of $\tilde\Lambda_{\bar\mu}$. Let γ_ϵ be the cut-off function in Lemma 6.10 Then $\gamma_\epsilon u$ has its support away from the singular set $\mathcal{S}$ and

$$\lim_{\epsilon\to 0}\int_M |\nabla(\gamma_\epsilon u)|^2\, \omega_\infty^n \,=\, \bar\mu \quad\text{and}\quad \lim_{\epsilon\to 0}\int_M (\gamma_\epsilon u)^2\, \omega_\infty^n \,=\, 1.$$

Recall that $\mathcal{S}$ is the singular set of ω_∞. Define $T_\delta(\mathcal{S})$ as the set of points in $\mathbb{C}P^N$ whose distance from $\mathcal{S}$ is less than δ. Then there are $\delta_i \to 0$ and diffeomorphisms $\phi_i : M_\infty \backslash T_{\delta_i}(\mathcal{S}) \mapsto M$ such that $\phi_i^*\omega_i$ converge to ω_∞ on $M_\infty\backslash\mathcal{S}$. It follows that there are $\epsilon_i \to 0$ such that $u_i = (\gamma_{\epsilon_i} u)\circ\phi_i^{-1}$ extend smoothly to M and

$$\lim_{i\to\infty}\int_{M_\infty} |\nabla u_i|^2\, \tilde\omega_i^n \,=\, \bar\mu \quad\text{and}\quad \lim_{i\to\infty}\int_{M_\infty} u_i^2\, \tilde\omega_i^n \,=\, 1.$$

For each a, there are eigenfuctions $u_{a,i}$ of ω_i which converge to u_a, then for each i sufficiently large, $u_i, u_{1,i}, \cdots, u_{k,i}$ generate a space Λ_i of dimension $k+1$ such that

$$\sup_{v\in\Lambda_i\backslash\{0\}} \frac{\int_M |\nabla v|^2\, \tilde\omega_i^n}{\int_M v^2\, \tilde\omega_i^n} \,\le\, \bar\mu \,+\, \nu_i$$

for some $\nu_i \to 0$. It follows that there are eigenfunctions $u_{0,i}$ with eigenvalue not bigger than $\bar\mu + \nu_i$ such that it has L^2-norm 1 and is orthogonal to $u_{a,i}$ for $a = 1, \cdots, k$. By using the Bochner technique, we have the eigenvalue for $u_{0,i}$ is not less than μ_i. By taking a subsequence if necessary, we may assume that $u_{0,i}$ converge to an eigenfunction $u_0 \neq 0$ in $\tilde\Lambda_{\bar\mu}$ which is orthogonal to u_a for $a = 1, \cdots, k$. A contradiction! Therefore, our claim is proved.

Let θ be a bounded function satisfying (6.31). By the above claim, it is the limit of eigenfunctions θ_i on M such that $\Delta_i\theta_i \,=\, -\lambda_i\,\theta_i$ with $\lim\lambda_i = \bar\mu$. Applying the Bochner identity, we get

$$\int_M |\nabla^{0,1}\bar\partial\theta_i|^2\, \tilde\omega_i^n \,\le\, (\lambda_i \,-\, \mu_i)\int_M |\partial\theta_i|^2\, \tilde\omega_i^n \,=\, \lambda_i(\lambda_i \,-\, \mu_i),$$

where $\nabla^{0,1}$ denotes the (0,1)-part of the covariant derivative of $\tilde{\omega}_i$. It follows that $\nabla^{0,1}\bar{\partial}\theta = 0$, so $\bar{\partial}\theta$ induces a holomorphic vector field Z outside the singular part $\mathcal{S}$ of $(M_\infty, \omega_\infty)$. If θ is real, then the imaginary part Y of Z is a Killing field. Since (6.31) is a real equation, we conclude that η_∞ is the complexication of a Lie algebra of Killing fields.

Finally, we want to extend Z to the ambient space $\mathbb{C}P^N$. This can be done as follows: It suffices to extend Y. Fix a small $\epsilon > 0$ such that $\overline{T_\epsilon(\mathcal{S})}$ is covered by finitely many open subsets $V_1, \cdots, V_k$ satisfying: (1) V_i is isomorphic to a ball in $\mathbb{C}^N$ and (2) For each i, there is a section σ_i in $H^0(M_\infty, K_{M_\infty}^{-\ell})$ such that $c \le ||\sigma_i||_\infty \le c^{-1}$ on $M_\infty \cap V_i$ for some $c > 0$ independent of i. We integrate Y to get a family of biholomorphic maps $\phi(t)$ from a neighborhood of $\overline{M_\infty \backslash T_\epsilon(\mathcal{S})}$ into $M_\infty \backslash \mathcal{S}$, where $|t| < \delta$ for some $\delta = \delta(\epsilon) > 0$. Note that $\phi(0) = I$. Since Y is a Killing field, wherever $\phi(t)$ is well-defined, it is an isometry of the induced Hermitian metric H_∞ on $K_{M_\infty}^{-1}$. Given any $\sigma \in H^0(M_\infty, K_{M_\infty}^{-\ell})$, $\phi(t)^*\sigma$ is a bounded and holomorphic section of $K_{M_\infty}^{-\ell}$ over $M_\infty \backslash T_\epsilon(\mathcal{S})$. If E is any subspace of $\mathbb{C}P^N$ of complex dimension $N - n + 2$ with (at most) finite intersections with $\mathcal{S}$, then $M_E = M_\infty \cap E$ is a complex normal variety of complex dimension 2 and $M_E \cap T_\epsilon(\mathcal{S})$ is compact. For each i, $f_i = \phi(t)^*\sigma/\sigma_i$ is a bounded holomorphic function on $(M_\infty \backslash T_\epsilon(\mathcal{S})) \cap V_i$, so by the Hartogs' extension theorem, f_i extends to be a bounded holomorphic function on $M_E \cap V_i$. It follows that $\phi(t)^*\sigma$ extends to a holomorphic section of $K_{M_\infty}^{-\ell}$ over $M_E \backslash \mathcal{S}$. Since E is arbitrary, we can easily deduce that $\phi(t)^*\sigma$ extends to $M_\infty \backslash \mathcal{S}$. Thus, $\phi(t)$ lifts to an isomorphism of $H^0(M_\infty, K_{M_\infty}^{-\ell})$, or equivalently, $\phi(t)$ is the restriction of an automorphism in $\mathbf{G} = \mathbf{SL}(N+1, \mathbb{C})$. Differentiating $\phi(t)$ on t, we see that Y, consequently Z, extends to a holomorphic vector field on $\mathbb{C}P^N$.[33]

Hence, η_∞ is reductive, and consequently, this lemma is proved.

□

By Lemma 6.9 and a known result in algebraic geometry (cf. [Do10]), we can find a $\mathbb{C}^*$-subgroup $\mathbf{G}_0 \subset \mathbf{G}$ which degenerates (M, D) to (M_∞, D_∞). Then we will get a contradiction to the K-stability as follows: Let X be the generating field of $\mathbf{G}_0$. As observed in [Do11] and [Li15]), adapting arguments from [Fu83], we can define the Futaki invariant $f_{M_\infty,(1-\beta)D_\infty}$, also referred as the log-Futaki invariant, for conic Kähler metrics on M_∞ with cone angle $2\pi\beta$ along D_∞ ($\bar{\beta} \in (0,1)$). Furthermore, if there is a conic Kähler-Einstein metric with angle $2\pi\beta$ along D_∞, then $f_{M_\infty,(1-\bar{\beta})D_\infty}$ vanishes. Note that though ω_∞ may not be smooth along D_∞ in our case, we can still prove the vanishing of $f_{M_\infty,(1-\bar{\beta})D_\infty}$ by adapting the arguments from [DT92]. It follows directly from the definition of the twisted Mabuchi energy (see [LS14]):

$$(\bar{\beta} - \bar{\beta}_1)\, \mathbf{M}_{\omega_0}(\psi_\tau) \;=\; (1 - \bar{\beta}_1)\, \mathbf{M}_{\omega_0,\bar{\mu}}(\psi_\tau) \;-\; (1 - \bar{\beta})\, \mathbf{M}_{\omega_0,\bar{\mu}_1}(\psi_\tau),$$

where $1 - \lambda^{-1} < \bar{\beta}_1 \le \bar{\beta}$ and ψ_τ is given in (6.4) for $\tau \in \mathbf{G}_0$. Taking derivative

[33]If $\beta_\infty < 1$, D_∞, being the singular set of ω_∞, must be preserved by the isometries generated by Y. Hence, Z is tangent to D_∞.

in τ and letting τ go to ∞, we get

$$(\bar{\beta} - \bar{\beta}_1)\, f_{M_\infty}(X) \;=\; -\,(1 - \bar{\beta})\, f_{M_\infty,(1-\bar{\beta}_1)D_\infty}(X).$$

There is a corresponding conic Kähler-Einstein metric with angle $2\pi\bar{\beta}_1$ if $\bar{\beta}_1$ is sufficiently close to $1 - \lambda^{-1}$ (see [LS14]). So $\mathbf{M}_{\omega_0,\bar{\mu}_1}$ is proper and consequently,

$$\mathrm{Re}(f_{M_\infty,(1-\beta_1)D_\infty}(X)) \;\geq\; 0.$$

Hence, we get

$$\mathrm{Re}(f_{M_\infty}(X)) \;\leq\; 0.$$

This contradicts to the assumption that M is K-stable and not biholomorphic to M_∞. Thus $\bar{\beta} \in E$ and Theorem 1.1 is proved.

We end up this section with a proof of Lemma 6.10. Put

$$\mathbf{I}_k \;=\; \int_{M_\infty} \sqrt{-1}\,\partial F \wedge \bar{\partial} F \wedge \omega_1 \wedge \cdots \wedge \omega_k \wedge \left(\frac{1}{\ell}\,\omega_{FS}\right)^{n-k-1}, \qquad (6.32)$$

where $F \;=\; \log(-\log ||\tau_\infty||_0^2)$ and $k = 0, \cdots, n-1$.

First we observe

$$\mathbf{I}_0 \;=\; \int_{M_\infty} \sqrt{-1}\,\partial F \wedge \bar{\partial} F \wedge \left(\frac{1}{\ell}\,\omega_{FS}\right)^{n-1} \;<\; \infty. \qquad (6.33)$$

To see this, we compute

$$\partial F \;=\; \frac{D\tau_\infty}{\tau_\infty\,(-\log ||\tau_\infty||_0^2)},$$

where D denotes the covariant derivative. Hence, we can write

$$\ell^{n-1}\,\mathbf{I}_0 \;=\; \int_{M_\infty} \frac{\sqrt{-1}\,D\tau_\infty \wedge \overline{D\tau_\infty}}{|\tau_\infty|^2\,(-\log ||\tau_\infty||_0^2)^2} \wedge \omega_{FS}^{n-1}.$$

It is known that the integral on the right is finite and is because the Poincaré metric on the punctured disc has finite volume. To see it, we choose a resolution $\pi : \tilde{M} \mapsto M_\infty$ such that $\pi^{-1}(\tau^{-1}(0))$ supports in a normal-crossing divisor with irreducible components D_a $(a = 1, \cdots, m)$. As usual, we denote by $[D_a]$ the line bundle corresponding to D_a. Then we have sections f_a of $[D_a]$ such that $D_a = \{f_a = 0\} \subset \tilde{M}$ and $\tau_\infty \circ \pi = f_1^{k_1} \cdots f_m^{k_m}$, where $k_a \geq 1$ are multiplicities of $\tau \circ \pi$ along D_a. Furthermore, we can have Hermitian norms $|| \cdot ||_a$ for $[D_a]$ such that for some constant $c > 0$,

$$|\mathfrak{f}_a||_a \;<\; 1 \quad \text{and} \quad \pi^* ||\tau_\infty||_0^2 \;=\; c \prod_{a=1}^m ||f_a||_a^{2k_a} \quad \text{on} \quad \tilde{M}.$$

Without loss of generality, we may assume that $c = 1$. [34] Then we have

$$\pi^*\left(\frac{\sqrt{-1}\,D\tau_\infty\wedge\overline{D\tau_\infty}}{|\tau_\infty|^2\,(-\log||\tau_\infty||_0^2)^2}\right) = \frac{\sqrt{-1}\,(\sum_a k_a\, f_a^{k_a-1}\,Df_a)\wedge(\sum_a k_a\,\overline{f_a\,Df_a})}{\prod_a|f_a|^{2k_a}\,(-\sum_a k_a\,\log||f_a||_a^2)^2}.$$

Using the Cauchy-Schwartz inequality and the fact that $-\log||f_a||_a^2 > 0$, we can deduce from the above

$$\int_{M_\infty}\frac{\sqrt{-1}\,D\tau_\infty\wedge\overline{D\tau_\infty}}{|\tau_\infty|^2\,(-\log||\tau_\infty||_0^2)^2}\wedge\omega_{FS}^{n-1}\;\le\; m\sum_{a=1}^m\int_{\tilde M}\frac{\sqrt{-1}\,Df_a\wedge\overline{Df_a}}{|f_a|^2\,(-\log||f_a||_a^2)^2}\wedge\pi^*\omega_{FS}^{n-1}.$$

A straightforward computation shows

$$\int_{\tilde M}\frac{\sqrt{-1}\,Df_a\wedge\overline{Df_a}}{|f_a|^2\,(-\log||f_a||_a^2)^2}\wedge\pi^*\omega_{FS}^{n-1}\;<\;\infty,\quad a\,=\,1,\cdots,m.$$

Then (6.33) follows.

We will prove Lemma 6.10 by induction. Suppose that we have proved for $i < k$

$$\mathbf{I}_i\;<\;\infty\quad\text{and}\quad\mathbf{V}_i\;=\;\mathbf{V}_0, \tag{6.34}$$

where

$$\mathbf{V}_i\;=\;\int_{M_\infty}\omega_1\wedge\cdots\wedge\omega_i\wedge\left(\frac{1}{\ell}\,\omega_{FS}\right)^{n-i}.$$

We need to prove (6.34) for k. Using the induction assumption, we have

$$\begin{aligned}&\mathbf{V}_k\;-\mathbf{V}_{k-1}\\ =\;&\lim_{\epsilon\to0}\int_{M_\infty}\gamma_\epsilon\,\omega_1\wedge\cdots\wedge\omega_{k-1}\wedge\sqrt{-1}\,\partial\bar\partial\,\psi_k\wedge\left(\frac{1}{\ell}\,\omega_{FS}\right)^{n-k}\\ =\;&\lim_{\epsilon\to0}\int_{M_\infty}\psi_k\,\sqrt{-1}\,\partial\bar\partial\gamma_\epsilon\wedge\omega_1\wedge\cdots\wedge\omega_{k-1}\wedge\left(\frac{1}{\ell}\,\omega_{FS}\right)^{n-k}\end{aligned}$$

By a direct computation, we have

$$\sqrt{-1}\,\partial\bar\partial\,\gamma_\epsilon\;=\;\frac{k\,\epsilon\,\tilde\eta'\,\omega_{FS}}{-\log||\tau_\infty||_0^2}\;+\;\left(-\,\epsilon\tilde\eta'\;+\;\epsilon^2\tilde\eta''\right)\,\sqrt{-1}\,\partial F\wedge\bar\partial F. \tag{6.35}$$

Since ψ_k is bounded, we can deduce from this

$$\left|\int_{M_\infty}\psi_k\,\sqrt{-1}\,\partial\bar\partial\gamma_\epsilon\wedge\omega_1\wedge\cdots\wedge\omega_{k-1}\wedge\omega_{FS}^{n-k}\right|\;\le\;C\,\epsilon\,(\mathbf{V}_{k-1}\;+\;\mathbf{I}_{k-1}).$$

Thus the integral on the left-handed side above converges to 0 as ϵ goes to 0, so we have $\mathbf{V}_k\;=\;\mathbf{V}_{k-1}\;=\;\mathbf{V}_0\;<\;\infty$.

[34] It is clear from the definition in (6.32) that $\mathbf{I}_0$ being finite or not won't be changed by replacing $||\cdot||_0$ with an equivalent norm.

It follows from (6.32) that

$$\mathbf{I}_k \;=\; \mathbf{I}_{k-1} \;-\; \lim_{\epsilon\to 0} \int_{M_\infty} \gamma_\epsilon\, \partial F \wedge \bar\partial F \wedge \omega_1 \wedge \cdots \wedge \omega_{k-1} \wedge \partial\bar\partial\, \psi_k \wedge \left(\frac{1}{\ell}\,\omega_{FS}\right)^{n-k-1}.$$

Denote by $\tilde{\mathbf{I}}(\epsilon)$ the second term on the right-handed side above, then

$$\tilde{\mathbf{I}}(\epsilon) \;=\; \int_{M_\infty} \psi_k\, \partial\bar\partial \left(\gamma_\epsilon\, \partial F \wedge \bar\partial F\right) \wedge \omega_1 \wedge \cdots \wedge \omega_{k-1} \wedge \left(\frac{1}{\ell}\,\omega_{FS}\right)^{n-k-1}. \qquad (6.36)$$

A straightforward computation yields

$$\partial\bar\partial\, \left(\gamma_\epsilon\, \partial F \wedge \bar\partial F\right) \;=\; -\,\epsilon\, \tilde\eta'\, \partial F \wedge \bar\partial F \wedge \partial\bar\partial F \;-\; \gamma_\epsilon\, \partial\bar\partial F \wedge \partial\bar\partial F.$$

Note that $\ell' = k\,\ell$ and

$$\sqrt{-1}\,\partial\bar\partial F \;=\; \frac{k\,\omega_{FS}}{-\log ||\tau_\infty||_0^2} \;-\; \sqrt{-1}\,\partial F \wedge \bar\partial F.$$

It follows

$$\partial\bar\partial\, \left(\gamma_\epsilon\, \partial F \wedge \bar\partial F\right) \;=\; \frac{k\,(\epsilon\tilde\eta' - 2\gamma_\epsilon)\,\sqrt{-1}\,\partial F \wedge \bar\partial F \wedge \omega_{FS}}{-\log ||\tau_\infty||_0^2} \;+\; \frac{\gamma_\epsilon\, k^2\, \omega_{FS} \wedge \omega_{FS}}{(-\log ||\tau_\infty||_0^2)^2}.$$

Thus, by the induction assumption, we can derive

$$\lim_{\epsilon\to 0} \tilde{\mathbf{I}}(\epsilon) \;=\; 2\,\ell'\, \tilde{\mathbf{I}}_{k,1} \;-\; (\ell')^2\, \tilde{\mathbf{I}}_{k,2},$$

where

$$\begin{aligned}
\tilde{\mathbf{I}}_{k,1} \;&=\; \int_{M_\infty} \frac{\psi_k}{-\log ||\tau_\infty||_0^2}\, \sqrt{-1}\,\partial F \wedge \bar\partial F \wedge \omega_1 \wedge \cdots \wedge \omega_{k-1} \wedge \left(\frac{1}{\ell}\,\omega_{FS}\right)^{n-k} \\
\tilde{\mathbf{I}}_{k,2} \;&=\; \int_{M_\infty} \frac{\psi_k}{(-\log ||\tau_\infty||_0^2)^2}\, \omega_1 \wedge \cdots \wedge \omega_{k-1} \wedge \left(\frac{1}{\ell}\,\omega_{FS}\right)^{n-k+1}.
\end{aligned}$$

Since ψ_k is bounded, we have for some constant $C > 0$

$$|\tilde{\mathbf{I}}_{k,1}| \;\le\; C\,\mathbf{I}_{k-1} \quad \text{and} \quad |\tilde{\mathbf{I}}_{k,2}| \;\le\; C\,\mathbf{V}_{k-1}.$$

Hence, by the induction assumption,

$$\mathbf{I}_k \;=\; \mathbf{I}_{k-1} \;+\; 2\,\ell'\, \mathbf{I}_{k,1} \;-\; (\ell')^2\, \mathbf{I}_{k,2} \;<\; \infty.$$

Note that $\partial\gamma_\epsilon = \epsilon\, \tilde\eta'\, \partial F$ and $|\tilde\eta'| \le 1$, then, in view of (6.32), we have

$$\int_{M_\infty} \sqrt{-1}\,\partial\gamma_\epsilon \wedge \bar\partial\gamma_\epsilon \wedge \omega_1 \wedge \cdots \wedge \omega_k \wedge \left(\frac{1}{\ell}\,\omega_{FS}\right)^{n-k-1} \;\le\; \epsilon^2\, \mathbf{I}_k.$$

Since $\mathbf{I}_k$ is finite, we see that the integral on the left-handed side above tends to 0 as ϵ goes to 0, and consequently, Lemma 6.10 is proved.

7 Appendix 1: The proof of Lemma 5.8

In this appendix, we complete the proof of Lemma 5.8. We will adopt the notations in Section 5, particularly, in the proof of the simple case of Lemma 5.8. If $\beta_\infty = 1$, then there is nothing to be proved since the singular set $\mathcal{S}_x$ is of complex dimension at least 2. So we may assume that $\beta_\infty < 1$. By using (2) of Lemma 5.5, we get a decomposition $\mathcal{S}_x = \mathcal{S}_x^0 \cup \bar{\mathcal{S}}_x$ satisfying: $\bar{\mathcal{S}}_x$ is a subcone of complex codimension at least 2 and any $y \in \mathcal{S}_x^0$ admits a tangent cone $\mathcal{C}_y$ of the form $\mathbb{C}^{n-1} \times \mathcal{C}'_y$. Furthermore, $\mathcal{C}'_y$ is the standard 2-dimensional cone with angle $2\pi\bar{\beta}$, so Lemma 5.8 has been proved for such a $\mathcal{C}_y$.

First we have $x_i \in M$ and $r_i > 0$ such that $(M, r_i^{-2}\omega_i, x_i)$ converge to the cone $(\mathcal{C}_x, g_x, o)$. This implies that there are $\lim \delta_i = 0$ and diffeomorphisms

$$\tilde{\phi}_i : V(x;\delta_i) \mapsto M\backslash T_{\delta_i}(D), \qquad T_{\delta_i}(D) = \{\, z \,|\, d_i(z, D) \le \delta_i \,\}, \tag{7.1}$$

where $d_i(\cdot, D)$ denotes the distance from D with respect to $r_i^{-2}\omega_i$, satisfying:

$$||r_i^{-2}\,\tilde{\phi}_i^*\omega_i - \omega_x||_{C^6(V(x;\delta_i))} \le \delta_i. \tag{7.2}$$

We may further assume that $\ell_i = r_i^{-2}$ are integers.

Secondly, by our assumption, there are integers $k_j = s_j^{-2}$ such that $(\mathcal{C}_x, k_j g_x, y)$ converge to $(\mathcal{C}_y, g_{\bar{\beta}}, o)$, where $\mathcal{C}_y = \mathbb{C}^{n-1} \times \mathcal{C}'_y$ with the flat conic metric $g_{\bar{\beta}}$ in the case of Lemma 5.8 already considered. Therefore, there are diffeomorphisms

$$\vartheta_j : V(y; j^{-1}) \subset \mathcal{C}_y \mapsto \mathcal{C}_x\backslash\mathcal{S}_x \tag{7.3}$$

satisfying:

$$||s_j^{-2}\vartheta_j^*\omega_x - \omega_{\bar{\beta}}||_{C^6(V(y;j^{-1}))} \le \frac{1}{j}, \tag{7.4}$$

where $\omega_{\bar{\beta}}$ is the Kähler form of $g_{\bar{\beta}}$.

It follows from (7.1) to (7.4) that for any $\delta > 0$, there are j_δ and i_δ such that for $j \ge j_\delta$ and $i \ge i_\delta$, we have $\tilde{\phi}_i \cdot \vartheta_j : V(y; j^{-1}) \mapsto M\backslash T_{\delta_i}(D)$ satisfying:

$$||k_j \ell_i\, \vartheta_j^* \tilde{\phi}_i^* \omega_i - \omega_{\bar{\beta}}||_{C^6(V(y;j^{-1}))} \le \delta. \tag{7.5}$$

Consider $\mathcal{C}_y \times \mathbb{C}$ as a bundle over $\mathcal{C}_y$ with the norm $e^{-|z'|^2 - |z_n|^{2\bar{\beta}}}\,|\cdot|^2$. Any holomorphic function f on $\mathbb{C}^n$ can be regarded as its section since $\mathcal{C}_y$ is biholomorphic to $\mathbb{C}^n$. Set $f_0 = \alpha_0, f_1 = \alpha_1 z_1, \cdots, f_n = \alpha_1 z_n$, where $\alpha_k > 0$ $(k = 0, \cdots, n)$ are chosen such that

$$\int_{\mathbb{C}^{n-1}\times\mathcal{C}'_y} |f_k|^2\, e^{-\left(|z'|^2 + |z_n|^{2\bar{\beta}}\right)}\, \omega_{\bar{\beta}}^n = 1. \tag{7.6}$$

Clearly, α_k are uniformly bounded. Applying Lemma 5.7 to each f_k,[35] for j and i sufficiently large, we get an isomorphism $\psi_{i,j}$ from $\mathcal{C}_x \times \mathbb{C}$ onto $K_M^{-k_j\ell_i}$ over $V(y; j^{-1})$ satisfying:

$$||\psi_{i,j}(f_k)||^2 = |f_k|^2\, e^{-\left(|z'|^2 + |z_n|^{2\bar{\beta}}\right)} \quad \text{and} \quad ||\nabla\psi_{i,j}||_{C^4(V(y;j^{-1}))} \le \delta. \tag{7.7}$$

[35] Here we replace M_∞ by M and ϕ by $\tilde{\phi}_i \cdot \vartheta_j$. In fact, we only need an easy case for Lemma 5.7 since the tangent cone $\mathcal{C}_y$ is simple.

Note that Lemma 5.8 has been proved for simple cones like $\mathcal{C}_y$, so we can apply the arguments for proving the partial C^0-estimate in Section 5 to construct holomorphic sections $S^k_{i,j}$ of $K_M^{-k_j\ell_i}$ over M such that

$$\sup_{V(y;j^{-1})\cap B_{10}(o,g_{\bar\beta})} \left|(\psi_{i,j})^*(S^k_{i,j}) \,-\, f_k\right| \,\le\, \frac{\epsilon}{2}, \tag{7.8}$$

where ϵ can be as small as we want so long as δ is sufficiently small. Moreover, by Corollary 4.2, we have

$$||\nabla S^k_{i,j}||_i \,\le\, C, \tag{7.9}$$

where $||\cdot||_i$ denotes the Hermitian norm associated to ω_i. Note that C always denotes a uniform constant. Hence, for certain $c>0$ depending only on α_0, we have

$$||S^0_{i,j}||_i \,\ge\, c \quad \text{on} \quad \tilde\phi_i(\vartheta_j(B_{10}(o,g_{\bar\beta}))). \tag{7.10}$$

Define a holomorphic map $F_{i,j} : \tilde\phi_i(\vartheta_j(B_{10}(o,g_{\bar\beta}))) \,\mapsto\, \mathbb{C}^n$ by

$$F_{i,j} \,=\, \left(\frac{S^1_{i,j}(x)}{S^0_{i,j}(x)},\cdots,\frac{S^n_{i,j}(x)}{S^0_{i,j}(x)}\right). \tag{7.11}$$

Then $F_{i,j}\cdot\tilde\phi_i\cdot\vartheta_j$ converge to the map $(f_1/f_0,\cdots,f_n/f_0)$ and in the smooth topology outside the singular set $\{(z',0)\}$ as $j,i\to\infty$, therefore, by taking j and i sufficiently large if necessary, we have

$$\left|F_{i,j}(\tilde\phi_i(\vartheta_j(z))) \,-\, (f_1/f_0,\cdots,f_n/f_0)\,(z)\right| \,\le\, \epsilon, \quad \forall\, z\in U_j, \tag{7.12}$$

where

$$U_j \,=\, \{(z',z_n)\in B_{10}(o,g_{\bar\beta}) \,|\, |z_n|^{\bar\beta} \,>\, j^{-1}\,\} \,\subset\, V(y;j^{-1}).$$

We may assume $B_{8s_jr_i}(x_i,\omega_i)\,\subset\,\tilde\phi_i(\vartheta_j(B_{10}(o,g_{\bar\beta})))$. It follows from (7.12) that for ϵ sufficiently small and i sufficiently large, $F_{i,j}$ is a holomorphic map from $B_{8s_jr_i}(x_i,\omega_i)$ onto its image which contains $B_{8-2\epsilon}(o,g_{\bar\beta})$.

By the derivative estimate (7.9), we get

$$\sup_{B_{8s_jr_i}(x_i,\omega_i)} ||dF_{i,j}||_{\omega_i} \,\le\, C\,(s_j\,r_i)^{-2}. \tag{7.13}$$

This is equivalent to

$$F^*_{i,j}\,\omega_0 \,\le\, C\,(s_j\,r_i)^{-2}\,\omega_i, \tag{7.14}$$

where ω_0 denotes the Euclidean metric on $\mathbb{C}^n$.

Next we claim: For j sufficiently large, $F_{i,j}(D\cap B_{7s_jr_i}(x_i,\omega_i))$ converge to a local divisor $D_j\,\subset\,\mathbb{C}^n$ as i goes to ∞. It will be proved by applying the Bishop theorem. For this purpose, we need to bound the volume of $F_{i,j}(D\cap B_{7s_jr_i}(x_i,\omega_i))$. Since $(\mathcal{C}_x, s_j^{-2}g_x, y)$ converge to the standard cone $\mathbb{C}^{n-1}\times\mathcal{C}'_y$ with the metric $g_{\bar\beta}$, for j and i sufficiently large, $F_{i,j}$ maps $D\cap B_{8s_jr_i}(x_i,\omega_i)$ into a tubular neighborhood:

$$T_{8,\epsilon} \,=\, \{(z',z_n)\,|\,|z'|<8, |z_n|<\epsilon\}.$$

This implies that the intersection of complex line segments $\{(z', z_n) \,|\, |z_n| \leq 6\}$ with $F_{i,j}(D \cap B_{8s_j r_i}(x_i, \omega_i))$ is independent of z' with $|z'| < 7.5$. Using the slicing argument in [CCT02] [36], one can show that for each z' with $|z'| < 7.5$, the complex line segment $\{(z', z_n) \,|\, |z_n| \leq 6\}$ intersects with $F_{i,j}(D \cap B_{8s_j r_i}(x_i, \omega_i))$ at exactly m points (counted with multiplicity), where $(1 - \bar{\beta}) = m\,(1 - \beta_\infty)$.

Let $\tilde{\eta} : \mathbb{R} \mapsto \mathbb{R}$ be a cut-off function satisfying: $\tilde{\eta}(t) = 1$ for $t \leq 56$, $\tilde{\eta}(t) = 0$ for $t > 60$, $|\tilde{\eta}'| \leq 1$ and $|\tilde{\eta}''| \leq 2$, then we have

$$\begin{aligned} & \int_{F_{i,j}(D \cap B_{8s_j r_i}(x_i, \omega_i))} \tilde{\eta}(|z'|^2)\, \omega_0^{n-1} \\ \leq & \int_{F_{i,j}(D \cap B_{8s_j r_i}(x_i, \omega_i))} (\tilde{\eta} + (n-1)\,|z_n|^2(\tilde{\eta}' \,+\, |z'|^2 \tilde{\eta}''))\,(dz' \wedge d\bar{z}')^{n-1}. \end{aligned}$$

It follows

$$\int_{F_{i,j}(D \cap B_{7.4 s_j r_i}(x_i, \omega_i))} \omega_0^{n-1} \;\leq\; 200\, nm. \tag{7.15}$$

Note that the limit of D coincides with $\mathcal{S}_x$ modulo a subset of Hausdorff codimension at least 4 under the Gromov-Hausdorff convergence of $(M, r_i^{-2}\omega_i, x_i)$ to $(\mathcal{C}_x, \omega_x, o)$.[37] It follows that $F_{\infty,j}(\mathcal{S}_x \cap B_{7s_j}(y, g_x))$ coincides with D_j.

The estimate (7.14) immediately implies

$$\int_{D \cap B_{6s_j r_i}(x_i, \omega_i)} F_{i,j}^* \omega_0^{n-1} \;\leq\; \int_{D \cap B_{6s_j r_i}(x_i, \omega_i)} (s_j r_i)^{-2n+2}\, \omega_i^{n-1}.$$

Applying the standard monotonicity to subvariety $F_{i,j}(D)$, we have

$$1 \;\leq\; \int_{F_{i,j}(D) \cap B_4(o, \omega_0)} \omega_0^{n-1} \;\leq\; \int_{D \cap B_{5s_j r_i}(x_i, \omega_i)} F_{i,j}^* \omega_0^{n-1}.$$

Then, letting i go to ∞, we can easily deduce from the above

$$s_j^{2n-2} \;\leq\; s_j^{2n-2} \int_{\mathcal{S}_x \cap B_{5s_j}(y, \omega_x)} F_{\infty,j}^* \omega_0^{n-1} \;\leq\; \mathbf{M}_{2n-2}(\mathcal{S}_x \cap B_{5s_j}(y, g_x)), \tag{7.16}$$

where $\mathbf{M}_{2n-2}$ denotes the $(2n-2)$-dimensional Hausdorff measure corresponding to g_x.

For convenience, we summarize the above as follows with one extra property.

Lemma 7.1. *For any $\epsilon > 0$ small, there is a j_ϵ such that for any $j \geq j_\epsilon$, the Lipschtz map $F_{\infty,j}$ maps $B_{7s_j}(y, g_x)$ into $B_{7+\epsilon}(o, g_{\bar{\beta}})$ satisfying:*

(1) Its image contains $B_{7-\epsilon}(o, g_{\bar{\beta}})$;

[36] We also refer the readers to the proof of Theorem 3.2, **C4**. In fact, it is easier here since for generic z', all the intersections are positive and transverse.

[37] Clearly, the limit lies in $\mathcal{S}_x$. On the other hand, by [CCT02], there is no singular point of $\mathcal{C}_x$ outside the limit of D for which there is a tangent cone of type $\mathbb{C}^{n-1} \times \mathcal{C}'_y$.

(2) $F_{\infty,j}(\mathcal{S}_x \cap B_{7s_j}(y,g_x))$ *is a local divisor* $D_j \subset T_{8,\epsilon}$*;*

(3) For any $\delta > 0$*, there is an* $\nu = \nu(\delta)$ *such that* $F_{\infty,j}^{-1}(T_{6,\nu}) \subset T_\delta(\mathcal{S}_x) \cap B_{7s_j}(y,g_x)$*.*

Proof. We have shown the validity of (1) and (2). For (3), we can prove by contradiction. If not true, then $F_{\infty,j}^{-1}(D_j \cap B_{6.5}(o,g_{\bar\beta}))$ has at least two distinct components, one lies in $\mathcal{S}_x$ while another is not. This implies that for i sufficiently large, the pre-image $F_{i,j}^{-1}(F_{i,j}(D) \cap B_{6.5}(o,g_{\bar\beta}))$ has at least two components. On the other hand, for j and i sufficiently large, when restricted to $B_{10s_jr_i}(x_i,\omega_i)\backslash T_\delta(D)$, $F_{i,j}$ is biholomorphic onto its image. It follows from (7.12) and (7.13) that $F_{i,j}(p) = F_{i,j}(p')$ for $p,p' \in B_{10s_jr_i}(x_i,\omega_i)$ only if $d_i(p,p')$ is sufficiently small. This implies that $F_{i,j}^{-1}(z)$ is either a point or a subvariety for any $z \in B_8(o,g_{\bar\beta})$. By (7.10), $B_{10s_jr_i}(x_i,\omega_i)$ lies in some $\mathbb{C}^{N'}$, where N' may depend on i,j. Thus $F_{i,j}$ is one-to-one on $B_{7s_jr_i}(x_i,\omega_i)$. This leads to a contradiction, so (3) is proved.

□

Next we observe: For i,j sufficiently large, there are uniformly bounded functions $\varphi_{i,j}$ on $B_{8s_jr_i}(x_i,\omega_i)$ satisfying:

$$(s_jr_i)^{-2}\omega_i = \sqrt{-1}\,\partial\bar\partial\,\varphi_{i,j} \quad \text{on } B_{8s_jr_i}(x_i,\omega_i). \tag{7.17}$$

This is because $||S_{i,j}^0||_i$ is close to an uniform constant for j,i sufficiently large. A consequence of this observation is that the volume of $D \cap B_{7s_jr_i}(x_i,\omega_i)$ with respect to $(s_jr_i)^{-2}\omega_i$ is uniformly bounded. To see this, we recall a well-known fact: If T is a positive, $\partial\bar\partial$-closed (1,1) current on $B_r(o,\omega_0)$, then for any bounded function φ on $B_r(o,\omega_0)$, we have

$$(T \llcorner B_{R-\kappa}(o,\omega_0))\,(\partial\bar\partial\varphi) \,\le\, C_\kappa \left(\sup_{B_R(o,\omega_0)} |\varphi|\right) T(\omega_0),$$

where C_κ depends only on κ. [38] For i,j sufficiently large, we have

$$F_{i,j}(B_{7s_jr_i}(x_i,\omega_i)) \subset B_{7+\epsilon}(o,\omega_0) \subset B_{7.4-\epsilon}(o,\omega_0) \subset F_{i,j}(B_{7.4s_jr_i}(x_i,\omega_i)).$$

May assume that $3n\,\epsilon < 1$. Put

$$T_a(\xi) = \int_{F_{i,j}(D)\cap B_{8-a\kappa}(o,\omega_0)} \xi \wedge \omega_i^{a-1} \wedge \omega_0^{n-a-1},$$

where $\kappa = \frac{1}{3n}$ and $a = 1,\cdots,n-1$. Then, applying the above fact to currents T_a and using (7.15), we get

$$\int_{D\cap B_{7s_jr_i}(x_i,\omega_i)} \omega_i^{n-1} \,\le\, C\,(s_jr_i)^{2n-2}.$$

[38] $T \llcorner U$ denotes the restriction to U and this can be easily proved by using a cut-off function and integration by parts.

Letting i go to ∞, we have

$$\mathbf{M}_{2n-2}(\mathcal{S}_x \cap B_{7s_j}(y, \omega_x)) \leq C\, s_j^{2n-2}. \tag{7.18}$$

For each $y \in \mathcal{S}_x^0$, we set $s(y) = s_j$ for a sufficiently large j such that Lemma 7.1 holds. Define U to be the union of all such balls $B_{6s(y)}(y, \omega_x)$, $\mathcal{S}'_x = \mathcal{S}_x \cap U$ and $\bar{\mathcal{S}}'_x = \mathcal{S}_x \backslash \mathcal{S}'_x$, then $\bar{\mathcal{S}}'_x$ is closed and contained in $\bar{\mathcal{S}}_x$. It follows from (7.18) and a simple covering argument that for any $R > 0$ and open neighborhood B of $\bar{\mathcal{S}}'_x$, there is a constant $C_{R,B}$ such that

$$\mathbf{M}_{2n-2}(\mathcal{S}_x \cap B_R(o, \omega_x) \backslash B) \leq C_{R,B}. \tag{7.19}$$

The following is the key lemma to our proof of Lemma 5.8.

Lemma 7.2. *We adopt the notations above. Assume that (1) $\xi : \mathbb{R} \mapsto [0,1]$ is a smooth function with $\xi(t) = 1$ for any $t \geq 8\epsilon$ and (2) f is a holomorphic function on $F_{\infty,j}(B_{7s_j}(y, g_x))$ such that $|f(z', z_n)| \geq |z_n|$ whenever $|z_n| \geq 8\epsilon$. Then there is a uniform constant C such that*

$$\begin{aligned} & s_j^{2-2n} \int_{B_{6s_j}(y,g_x)} |\nabla(h \cdot F_{\infty,j})|^2_{\omega_x}\, \omega_x^n \\ \leq\ & C \int_{F_{\infty,j}(B_{7s_j}(y,g_x))} \sqrt{-1}\,\partial h \wedge \bar{\partial} h \wedge (dz' \wedge d\bar{z}')^{n-1}, \end{aligned} \tag{7.20}$$

where $h(z', z_n) = \xi(|f|^2(z', z_n))$.

Proof. It suffices to prove the corresponding inequality for each $F_{i,j}$ and then let i go to ∞. As above, let $\tilde{\eta} : \mathbb{R} \mapsto \mathbb{R}$ be a cut-off function such that $\tilde{\eta}(t) = 1$ for $t \leq 40$, $\tilde{\eta}(t) = 0$ for $t > 46$, $|\tilde{\eta}'| \leq 1$ and $|\tilde{\eta}''| \leq 2$, then we have

$$\sqrt{-1}\,\partial\bar{\partial}\,\tilde{\eta}(|z'|^2) \leq 200\, n\, dz' \wedge d\bar{z}'.$$

By our assumptions (1) and (2), we can show that $\tilde{\eta}(|z'|^2)\,|dh|^2$ vanishes near the boundary of $F_{i,j}(B_{7s_jr_i}(x_i, \omega_i))$. It is easy to see

$$\partial h \wedge \partial\bar{\partial} h = 0.$$

Using these facts, (7.17) and integration by parts, we can deduce

$$\begin{aligned} & (s_j r_i)^{-2n} \int_{B_{7s_jr_i}(x_i,\omega_i)} \eta(|z'|^2)\, |\nabla(h \cdot F_{i,j})|^2_{\omega_i}\, \omega_i^n \\ =\ & n \int_{F_{i,j}(B_{7s_jr_i}(x_i,\omega_i))} \eta(|z'|^2)\, \sqrt{-1}\,\partial h \wedge \bar{\partial} h \wedge (\sqrt{-1}\,\partial\bar{\partial}(\varphi_{i,j} \cdot F_{i,j}^{-1}))^{n-1} \\ \leq\ & C \int_{F_{i,j}(B_{7s_jr_i}(x_i,\omega_i))} \sqrt{-1}\,\partial h \wedge \bar{\partial} h \wedge (dz' \wedge d\bar{z}')^{n-1}. \end{aligned}$$

Then the lemma follows by letting i go to ∞.

∎

Now we complete the proof of Lemma 5.8. Let $\bar\epsilon$ be given in Lemma 5.8.

Fix a small $\epsilon_0 > 0$, since $\bar{\mathcal{S}}'_x$ is closed and has vanishing Hausdorff measure of dimension strictly bigger than $2n-4$, we can find a finite cover of $\bar{\mathcal{S}}'_x \cap B_{\bar\epsilon^{-1}}(x, g_x)$ by balls $B_{r_a}(y_a, g_x)$ $(a = 1, \cdots, l)$ satisfying:

(i) $y_a \in \bar{\mathcal{S}}'_x$ and $2r_a \le \epsilon_0/l$;

(ii) $\sum_a r_a^{2n-3} \le 1$;

We denote by $\bar\eta$ a cut-off function: $\mathbb{R} \mapsto \mathbb{R}$ satisfying: $0 \le \bar\eta \le 1$, $|\bar\eta'(t)| \le 2$ and

$$\bar\eta(t) = 1 \text{ for } t > 1.6 \text{ and } \bar\eta(t) = 0 \text{ for } t \le 1.1.$$

Set $\chi = \prod_a \chi_a$, where

$$\chi_a(y) = \bar\eta\left(\frac{d(y, y_a)}{r_a}\right) \quad \text{if } y \in B_{2r_a}(y_a, g_x) \quad \text{and} \quad \chi_a(y) = 1 \text{ otherwise.}$$

Then χ vanishes on the closure of $B = \cup_a B_{r_a}(y_a, g_x)$ which contains $\bar{\mathcal{S}}'_x \cap B_{\bar\epsilon^{-1}}(x, g_x)$, furthermore, χ satisfies

$$\int_{\mathcal{C}_x} |\nabla\chi|^2\, \omega_x^n \le l \int_{B_{2r_a}(x_a, g_x)} |\nabla\chi_a|^2\, \omega_x^n \le C\, l \sum_a r_a^{2n-2} \le C\,\epsilon_0, \tag{7.21}$$

where C is a uniform constant.

There is a finite cover of $\mathcal{S}_x \cap B_{\bar\epsilon^{-1}}(x, g_x) \backslash B$ by balls $B_{6s_b}(y_b, g_x)$ for which Lemma 7.1 holds $(b = 1, \cdots, N)$. Choose smooth functions $\{\zeta_b\}$ associated to the cover $\{B_{6s_b}(y_b, g_x)\}$ satisfying:

(1) $0 \le \zeta_b \le 1, \quad |\nabla\zeta_b| \le s_b^{-1}$;

(2) $\mathrm{supp}(\zeta_b)$ is contained in $B_{6s_b}(y_b, g_x)$;

(3) $\sum_b \zeta_b \equiv 1$ near $\mathcal{S}_x \cap B_{\bar\epsilon^{-1}}(x, g_x) \backslash B$.

Then $\{\zeta_b\}, 1 - \sum_b \zeta_b$ form a partition of unit for the cover $\{B_{6s_b}(y_b, g_x)\}$ and $B_{\bar\epsilon^{-1}}(x, g_x)$.

As before, we denote by η a cut-off function: $\mathbb{R} \mapsto \mathbb{R}$ satisfying: $0 \le \eta \le 1$, $|\eta'(t)| \le 1$ and

$$\eta(t) = 0 \text{ for } t > \log(-\log\bar\delta^3) \text{ and } \eta(t) = 1 \text{ for } t < \log(-\log\bar\delta).$$

For each b, let F_b be the map from $B_{7s_b}(y_b, g_x)$ into $B_{7+\epsilon}(o, g_{\bar\beta})$ and $D_b \subset B_{7+\epsilon}(o, g_{\bar\beta_b})$ be the divisor given by Lemma 7.1. Let ν be given in (3) of Lemma 7.1 for any small δ. It is clear from its proof that ν can be chosen independent of b. May assume that $10\epsilon > \nu$ and f_b be a local defining function of D_b satisfying (2) in Lemma 7.2. We define a function $\gamma_{\bar\epsilon,b}$ on $B_{7s_b}(y_b, g_x)$ as follows: If $|f_b|(F_b(y)) \ge \bar\epsilon/3$, put $\gamma_{\bar\epsilon,b}(y) = 1$ and if $|f_b|(F_b(y)) < \bar\epsilon$, put

$$\gamma_{\bar\epsilon,b}(y) = \eta\left(\log\left(-\log\left(\frac{|f_b|(F_b(y))}{\bar\epsilon}\right)\right)\right). \tag{7.22}$$

For any ϵ_0', by choosing $\bar{\delta}$ sufficiently small, we can deduce from Lemma 7.2

$$\int_{B_{6s_b}(y_b,g_x)} |\nabla \gamma_{\bar{\epsilon},b}|^2 \,\omega_x^n \;\le\; \epsilon_0' \, s_b^{2n-2}. \tag{7.23}$$

Moreover, by (3) of Lemma 7.1, we may have $\gamma_{\bar{\epsilon},b}(y) = 1$ if $d(y, \mathcal{S}_x) \ge \epsilon_0'$. Now we define

$$\gamma_{\bar{\epsilon}}(y) \;=\; \chi(y)\,(1 - \sum_b \zeta_b(y) \,+\, \sum_b \zeta_b(y)\,\gamma_{\bar{\epsilon},b}(y)). \tag{7.24}$$

Then $\gamma_{\bar{\epsilon}}(y) = 1$ whenever y is outside B and $d(y, \mathcal{S}_x) \ge \epsilon_0'$. Also $\gamma_{\bar{\epsilon}}$ vanishes in a neighborhood of $\mathcal{S}_x$. It follows from (7.24) and (7.21)

$$\int_{B_{\bar{\epsilon}^{-1}}(o,g_x)} |\nabla \gamma_{\bar{\epsilon}}|^2 \,\omega_x^n \le C \left(\epsilon_0 + N \sum_b \int_{B_{6s_b}(y_b,g_x)} |\nabla(\zeta_b(1-\gamma_{\bar{\epsilon},b}))|^2 \,\omega_x^n \right).$$

By (7.23) and (7.16), we have

$$\int_{B_{6s_b}(y_b,g_x)} |\nabla(\zeta_b(1-\gamma_{\bar{\epsilon},b}))|^2 \,\omega_x^n \;\le\; 4\,\epsilon_0'\, s_b^{2n-2} \;\le\; 4\,\epsilon_0'\, \mathbf{M}_{2n-2}(\mathcal{S}_x \cap B_{5s_b}(y_b, g_x)).$$

Set $U' = \cup_b B_{5s_b}(y_b, g_x)$, then $B' = B \backslash \bar{U}'$ is a neighborhood of $\bar{\mathcal{S}}_x' \cap B_{\bar{\epsilon}^{-1}}(x, g_x)$. It follows from the above two estimates and (7.19)

$$\int_{B_{\bar{\epsilon}^{-1}}(o,g_x)} |\nabla \gamma_{\bar{\epsilon}}|^2 \,\omega_x^n \;\le\; C\left(\epsilon_0 + 4N^2\, C_{\bar{\epsilon}^{-1},B'}\,\epsilon_0'\right).$$

Thus, we can complete the proof of Lemma 5.8 by taking ϵ_0 and then ϵ_0' sufficiently small.

8 Appendix 2: A previous result of Tian-Wang

In this appendix, for the readers' convenience, we give an outlined proof of a result in [TW12] on almost Kähler-Einstein manifolds. This is needed for proving the partial C^0-estimate when cone angles $2\pi\beta$ converge to 2π. For simplicity, we need to consider only the following situation: [39] Let $(M, \tilde{\omega}_i)$ be a sequence of smooth Kähler manifolds with Kähler class $2\pi c_1(M)$ and satisfying:

$$\mathrm{Ric}(\tilde{\omega}_i) \;\ge\; \mu_i \tilde{\omega}_i, \quad \text{ where } \;\lim \mu_i \;=\; 1.$$

Theorem 8.1. *Assume that $(M, \tilde{\omega}_i)$ converge to (M_∞, d_∞) in the Gromov-Hausdorff topology. Then M_∞ is smooth outside a closed subset $\mathcal{S}$ of complex codimension at least 2 and the distance d_∞ is induced by a Kähler-Einstein metric ω_∞ on $M_\infty \backslash \mathcal{S}$. Moreover, any tangent cone $\mathcal{C}_x$ of M_∞ is Kähler-Ricci flat outside a closed subcone $\mathcal{S}_x$ of complex codimension at least 2.*

[39] All the results in this appendix are taken from [TW12]. I thank B. Wang for agreeing to my doing so.

The rest of this section is devoted to an outlined proof of this theorem. First we observe that the sequence $(M, \tilde\omega_i)$ is almost Kähler-Einstein in the sense of [TW12] since we have

$$\begin{aligned}\int_M |\mathrm{Ric}(\tilde\omega_i) - \tilde\omega_i|\,\tilde\omega_i^n &\le n\int_M (\mathrm{Ric}(\tilde\omega_i) - \mu_i\tilde\omega_i)\wedge\tilde\omega_i^{n-1} + (1-\mu_i)\int_M \tilde\omega_i^n \\ &= 2\,(1-\mu_i)\int_M \tilde\omega_i^n \to 0. \end{aligned} \tag{8.1}$$

The following is crucial and a special case of Proposition 3.1 in [TW12].

Proposition 8.2. *For any $\alpha, r \in (0,1]$, there are $\delta = \delta(n,\alpha)$ and $\epsilon = \epsilon(n,\alpha)$ with the property: If $(M, \omega(t))$ is a Ricci flow:*

$$\frac{\partial \omega(t)}{\partial t} = -\,\mathrm{Ric}(\omega(t)) \tag{8.2}$$

and satisfies:

$$\mathrm{Ric}(\omega(0)) \ge 0, \quad \text{and} \quad \frac{\mathrm{Vol}(B_r(x_0,\omega(0)))}{r^{2n}} \ge (1-\delta)\,c_n, \tag{8.3}$$

then for any $x \in B_{\epsilon r}(x_0, \omega(0))$ and $t \in (0, (\epsilon r)^2]$, we have

$$\mathrm{Vol}(B_{\sqrt t}(x,\omega(t))) \ge \kappa_n\, t^n, \quad |\mathrm{Rm}|(\omega(t)) \le \alpha t^{-1} + (\epsilon r)^{-2},$$

where κ_n is a uniform constant and Rm *denotes the curvature tensor.*

Proof. It suffices to prove the curvature estimate. The volume bound follows from this curvature estimate.

If this proposition is false, then we have sequences $\delta_k, \epsilon_k \to 0$, $(M_k, \omega_k(t))$ satisfying (8.2) and $x_k \in M_k$ such that (8.15) holds while the curvature estimate fails. By scaling, we may assume that $r = 1$.

Following the proof of Perelman's pseudo-locality theorem (Theorem 10.1 in [Pe02]), we can find $u_k > 0$ with compact support in $B_1(x_k, \omega_k(0))$ satisfying:

$$\int_{B_1(x_k,\omega_k(0))} u_k^2 = 1 \quad \text{abd} \quad \mathcal{F}(u_k) \le -\eta, \tag{8.4}$$

where

$$\mathcal{F}(u_k) = \int_{B_1(x_k,\omega_k(0))} \left\{2|\nabla u_k|^2 - 2u_k^2 \log u_k - 2n\left(1 + \log\sqrt{2\pi}\right) u_k^2\right\}. \tag{8.5}$$

Then by a result of Rothaus [RO81], we get a minimizer φ_k of $\mathcal{F}_k$ satisfying:

$$-\Delta\varphi_k - \varphi_k \log\varphi_k - n\left(1 + \log\sqrt{2\pi}\right)\varphi_k = \lambda_k\,\varphi_k. \tag{8.6}$$

Here $2\lambda_k = \mathcal{F}(\varphi_k) \le \mathcal{F}(\bar u_k) \le -\eta < 0$. By using the Sobolev inequality, we can easily show that $|\lambda_k| \le C$ for some uniform constant C. On the other hand, using (8.6) and the Moser iteration, we can bound $||\varphi_k||_{C^0}$. Since the Ricci curvature of $\omega_k(0)$ is positive, by the gradient estimate of Cheng-Yau, we have

$$|\nabla\varphi_k|(x) \le C_2(n, d(x, \partial B_1(x_k, \omega_k(0)))), \quad \text{where } x \in B_1(x_k, \omega_k(0)). \tag{8.7}$$

Without loss of generality, we may assume that $(M_k, \omega_k(0), x_k)$ converge to $(M_\infty, \omega_\infty, x_\infty)$ in the Gromov-Hausdorff topology. In fact, by the condition on volume ratios, $M_\infty = \mathbb{R}^{2n}$ and ω_∞ is the Euclidean metric. It follows from the above that by taking a subsequence if necessary, φ_k converge to a locally-Lipschtz function φ_∞ on $B_1(x_\infty, \omega_\infty) \subset M_\infty$.

Next, we show

Claim: φ_∞ can be extended to be a continuous function on $\overline{B_1(x_\infty, \omega_\infty)}$ with

$$\varphi_\infty|_{\partial B_1(x_\infty, \omega_\infty)} = 0. \tag{8.8}$$

It suffices to show that $\lim_{r\to 0} ||\varphi_\infty||_{L^\infty(B_r(z))} = 0$ for arbitrary $z \in \partial B_1(x_\infty)$. Here we denote by $B_r(z)$ the ball in M_∞ with center z and radius r.

For any $z \in \partial B_1(x_\infty)$. Suppose $z_k \in \partial B_1(x_k, \omega_k(0))$ and $\lim z_k = z$. Put

$$M_{d,k} = \sup_{B_d(z_k, \omega_k(0))} \varphi_k - \inf_{B_d(z_k, \omega_k(0))} \varphi_k, \quad \psi_{d,k} = M_{2d,k} - \varphi_k.$$

Note that by trivial extension, we can regard φ_k as defined on M_k. Using (8.6), in the sense of distribution, we have

$$\left(-\Delta - \left(n + n\log\sqrt{2\pi} + \lambda_k\right)\right)\psi_{d,k} \ge -C_3, \tag{8.9}$$

where C_3 is a uniform constant independent of k. Then, by the standard Moser iteration, we obtain

$$(2d)^{-2n} \int_{B_{2d}(z_k, \omega_k(0))} \psi_{d,k} \le C_4 \left(\inf_{B_d(z_k, \omega_k(0))} \psi_{d,k} + d^2\right). \tag{8.10}$$

It is not hard to see that

$$\mathrm{Vol}\left(B_{2d}(z_k, \omega_k(0)) \backslash B_1(x_k, \omega_k(0))\right) > c_n 10^{-n} (2d)^{2n}$$

and

$$\inf_{B_d(z_k, \omega_k(0))} \psi_{d,k} = M_{2d,k} - M_{d,k}.$$

Plugging these into (8.10), we get

$$10^{-n} c_n M_{2d,k} < C_4 \left(M_{2d,k} - M_{d,k} + d^2\right).$$

This implies for some $\gamma \in (0, 1)$,

$$M_{d,k} < \gamma M_{2d,k} + d^2. \tag{8.11}$$

Let $d = 2^{-i}$ $(i > 1)$, iteration of (8.11) on i yields

$$M_{2^{-i},k} < \gamma^{i-1} M_{\frac{1}{2},k} + \frac{\gamma^{i-1} - 4^{-i+1}}{4(4\gamma - 1)}.$$

Since $||\varphi_k||_{L^\infty(B_1(x_k,\omega_k(0)))} \le C_1$, letting $k \to \infty$, we obtain

$$||\varphi_\infty||_{L^\infty(B_{2^{-i}}(z))} \le C_1 \gamma^{i-1} + \frac{\gamma^{i-1} - 4^{-i+1}}{4(4\gamma - 1)}. \tag{8.12}$$

Then our **Claim** follows.

By the standard arguments, we can prove that on $B_1(x_\infty)$, φ_∞ satisfies

$$-\Delta\varphi_\infty - \varphi_\infty \log \varphi_\infty - \left(n + n \log \sqrt{2\pi} + \lambda_\infty\right) \varphi_\infty = 0. \tag{8.13}$$

Consequently, $\varphi_\infty \in C^\infty(B_1(x_\infty))$.

Now we can derive a contradiction. In fact, by trivial extension, we can regard $\varphi_\infty \in W_0^{1,2}(\mathbb{R}^{2n})$. Then, by (8.13), we have

$$\int_{\mathbb{R}^{2n}} \left(|\nabla\varphi_\infty|^2 - \varphi_\infty^2 \log \varphi_\infty - n \left(1 + \log \sqrt{2\pi}\right) \varphi_\infty^2\right) = \lambda_\infty \le -\eta < 0.$$

This contradicts to the Logarithmic Sobolev inequality for $\mathbb{R}^{2n}$ (cf. [Gro93]) which implies

$$\int_{\mathbb{R}^{2n}} \left(|\nabla\varphi_\infty|^2 - \varphi_\infty^2 \log \varphi_\infty - n \left(1 + \log \sqrt{2\pi}\right) \varphi_\infty^2\right) \ge 0.$$

Therefore, our proposition is proved. □

Corollary 8.3. *There is a $\delta = \delta(n)$ with the property: If $(M, \omega(t))$ is a normalized Ricci flow:*

$$\frac{\partial\omega(t)}{\partial t} = \lambda_0 \, \omega(t) - \mathrm{Ric}(\omega(t)) \tag{8.14}$$

where λ_0 is a constant with $|\lambda_0| \le 1$. Suppose that

$$\mathrm{Ric}(\omega(0)) \ge 0 \text{ on } B_1(x_0, \omega(0)) \text{ and } \mathrm{Vol}(B_1(x_0, \omega(0))) \ge (1 - \delta)\, c_n. \tag{8.15}$$

Then for any $x \in B_{\frac{3}{4}}(x_0, \omega(0))$ and $t \in (0, 2\delta]$, we have

$$\mathrm{Vol}(B_{\sqrt{t}}(x, \omega(t))) \ge \kappa_n \, t^n \text{ and } |\mathrm{Rm}|(\omega(t)) \le t^{-1}.$$

This follows from applying Proposition 8.2 to

$$\tilde\omega(t) = (1 - 2\lambda_0 t)\, \omega\Big(\frac{\log(1 - 2\lambda_0 t)}{-\lambda_0}\Big)$$

which is a solution of Ricci flow.

Next we estimate the change of distance function along (8.14).

Proposition 8.4 (Theorem 4.1 in [TW12]). *Let δ and $(M, \omega(t))$ be as in last corollary. Then for any $x_1, x_2 \in B_{\frac{1}{4}}(x_0, \omega(0))$, we have*

$$l - C_0 E^{\frac{1}{2(2n+3)}} \le d_{\omega(\delta)}(x_1, x_2) \le l + C\, l\, E^{\frac{1}{6n(2n+3)}}, \tag{8.16}$$

where $l = d_{\omega(0)}(x_1, x_2)$, C_0, C are uniform constants and E is defined as

$$E = E(\delta) = \int_0^{2\delta} \int_{B_{\frac{1}{2}}(x_0, \omega(0))} |R(\omega(t)) - n\lambda_0|\, \omega(t)^n \wedge dt.$$

Proof. We will sketch its proof and refer to [TW12] for more details. We will always denote by C_0, C uniform constants.

First we observe that for any $x \in B_{\frac{1}{2}}(x_0, \omega(0))$ and $s \in (0, \delta]$,

$$|\mathrm{Ric}(\omega(s)) - \lambda_0\, \omega(s)|(x) \le C\, s^{-(n+2)}\, E(s)^{\frac{1}{2}}. \tag{8.17}$$

This follows from the curvature estimates in Corollary 8.3 and applying Moser's iteration to the following curvature evolutions on $B_1(x_0, \omega(0))$:

$$\begin{aligned} \frac{\partial |h|}{\partial t} &\le \frac{1}{2}\Delta |h| + |Rm|\, |h|, \\ \frac{\partial H}{\partial t} &= \frac{1}{2}\Delta H + |h|^2 + \lambda_0 H, \end{aligned}$$

where $h(t) = \mathrm{Ric}(\omega(t)) - \lambda_0 \omega(t)$ and $H = \mathrm{tr}_{\omega(t)} h(t)$.

Secondly, we recall an estimate of R. Hamilton: If $\mathrm{Ric}(\omega(t))(x) \le K$ for any x in $B_r(x_i, \omega(t))$ $(i = 1, 2)$, then

$$\frac{\partial d_{\omega(t)}(x_1, x_2)}{\partial t} \ge \frac{\lambda_0}{2}\, d_{\omega(t)}(x_1, x_2) - (Kr + r^{-1}).$$

Let $t_1 \in (0, \delta]$ be the maximum of t such that $B_{\sqrt{t}}(x_i, \omega(t)) \subset B_{3/4}(x_0, \omega(0))$ $(i = 1, 2)$, then by Corollary 8.3, for any $t \in [0, t_1]$, we have

$$d_{\omega(t)}(x_1, x_2) \ge d_{\omega(0)}(x_1, x_2) - C_0 \sqrt{t}. \tag{8.18}$$

This implies that $t_1 = \delta$. On the other hand, by (8.17) and integrating along (8.14), we get

$$\left| \log \left(\frac{d_{\omega(\delta)}(x_1, x_2)}{d_{\omega(t_0)}(x_1, x_2)} \right) \right| \le C_0\, t_0^{-(n+1)}\, E^{\frac{1}{2}}. \tag{8.19}$$

Choosing t_0 at the order of $E^{\frac{1}{2n+3}}$, we can deduce the LHS of (8.16) from (8.18) and (8.19).

In the following, we show the RHS of (8.16). The idea is roughly as follows: The LHS says that the identity map is an almost expanding map, but it is also an almost volume preserving map if E is sufficiently small, so it should be also an almost isometry. Let us examine it more closely.

Assume that $B_{r_0}(x,\omega(0))$ be the largest geodesic ball which is contained in $B_l(x_1,\omega(0))$ and disjoint from $B_{l-\varepsilon}(x_1,\omega(\delta))$, where $\varepsilon = C_0 E^{\frac{1}{2(2n+3)}}$. By the volume comparison and the smallness of the change of volume along (8.14), we can deduce

$$\mathrm{Vol}_{\omega(\delta)}(B_{r_0}(x,\omega(0))) \,\le\, \mathrm{Vol}_{\omega(0)}(B_l(x,\omega(0))) \,-\, \mathrm{Vol}_{\omega(\delta)}(B_{l-\varepsilon}(x,\omega(\delta))) \,+\, E.$$

It follows that whenever $C\,E^{\frac{1}{2(2n+3)}} << l$,

$$r_0 \,\le\, \left(|V_l^{-1}-1| + C l^{-1}\varepsilon\right)^{\frac{1}{2n}}\, l \,+\, \varepsilon,$$

where

$$V_l \,=\, \inf\left\{ \frac{\mathrm{Vol}_{\omega(\delta)}(B_r(x,\omega(\delta)))}{r^{2n}} \,|\, B_r(x,\omega(\delta)) \subset B_{\frac{1}{2}}(x_0,\omega(0)),\ r\le l \right\}.$$

By the definition of r_0, we can find $x_3 \in B_{3r_0}(x_2,\omega(0)) \cap B_{l-\varepsilon}(x_1,\omega(\delta))$. Then we claim

$$d_{\omega(\delta)}(x_2,x_3) \,\le\, C\,r_1, \quad \text{where} \ \ r_1 \,=\, \max\{3\varepsilon, r_0\}.$$

This claim can be proved by a simple covering argument: Join x_2 to x_3 by minimal $\omega(0)$-geodesic γ and cover it by N balls $B_{2r_1}(z_i,\omega(\delta))$ such that $z_i\in\gamma$ and $B_{r_1}(z_i,\omega(\delta))$ are mutually disjoint. Clearly, all $B_{r_1}(z_i,\omega(\delta))$ are contained in $B_{5r_1}(x_2,\omega(0))$, so we have

$$N V_l\, r_1^{2n} \,\le\, \mathrm{Vol}_{\omega(\delta)}(\cup_i B_{r_1}(z_i,\omega(\delta))) \,\le\, \mathrm{Vol}_{\omega(0)}(B_{5r_1}(x_2,\omega(0))) \,+\, E.$$

Then N is bounded from above by $C(1+Er_1^{-2n})$ and consequently, our claim follows. It follows from the above claim and the estimate on r_0

$$d_{\omega(\delta)}(x_1,x_2) \,\le\, C\, \left(|V_l^{-1}-1|^{\frac{1}{2n}} \,+\, l^{\frac{1}{2n}}\, E^{\frac{1}{4n(2n+3)}}\right) l \tag{8.20}$$

whenever $E << l^{2(2n+3)}$.

Now we can conclude the proof. Since the curvature of $\omega(\delta)$ is bounded on $B_{3/4}(x,\omega(0)))$, there is a $\xi=\xi(n,\delta)$ such that $A_r \ge 1 - Cr^2$ for any $r\le\xi$. It follows from (8.20)

$$d_{\omega(\delta)}(y_1,y_2) \,\le\, r\,(1 \,+\, Cr^{\frac{1}{n}} \,+\, r^{\frac{1}{2n}}\, E^{\frac{1}{4n(2n+3)}}\,)$$

whenever $y_1,y_2\in B_{1/2}(x,\omega(0)))$ and $d_{\omega(0)}(y_1,y_2)\le r\le\xi$. Then our proposition follows from this by a simple covering argument. □

Now we return to the proof of Theorem 8.1. Consider the normalized Ricci flow with initial metric $\tilde\omega_i$ as in Theorem 8.1:

$$\frac{\partial\omega_i(t)}{\partial t} \,=\, \omega_i(t) \,-\, \mathrm{Ric}(\omega_i(t)), \quad \omega_i(0) \,=\, \tilde\omega_i. \tag{8.21}$$

Note that (8.2) has a unique solution $\omega_i(t)$ on $M \times [0,\infty)$. By using (8.1) and the estimate on the lower bound of scalar curvature $R(\omega_i(t))$ along the flow, we have

$$\int_0^1 \int_M |R(\omega_i(t)) \,-\, n|\,\omega_i(t)^n \wedge dt \,\le\, 2n(e-1)(1-\beta_i)\int_M \omega_i(t)^n \,\to\, 0. \quad (8.22)$$

Let $\mathcal{S}_k$ $(0 \le k \le 2n-1)$ denote the subset of M_∞ consisting of points for which no tangent cone splits off a factor, $\mathbb{R}^{k+1}$, isometrically. Clearly,

$$\mathcal{S}_0 \subset \mathcal{S}_1 \subset \cdots \subset \mathcal{S}_{2n-1}.$$

It is proved by Cheeger-Colding that $\mathcal{S}_{2n-1} = \emptyset$ and $\dim \mathcal{S}_k \le k$. For any $x \in M_\infty \backslash \mathcal{S} = \mathcal{S}_{2n-2}$, every tangent cone is $\mathbb{R}^{2n}$, so there is a $r > 0$ satisfying:

$$\mathrm{Vol}(B_r(x, d_\infty)) \,\ge\, (1-\delta)\, c_n,$$

where $\delta > 0$ is chosen smaller than the ones in Proposition 8.2 and 8.4. Then by using Proposition 8.4 and (8.22),, we know that $B_{r/4}(x, d_\infty)$ is also the Gromov-Hausdorff limit of $B_{r/4}(x_i, \omega_i(t))$ for some $x_i \in M$ and any $t \in (0, \delta]$. Since the curvature of $\omega_i(t)$ is uniformly bounded by t^{-1}, $\omega_i(t)$ converge to a smooth metric ω_∞ which induces d_∞ for any $\in (0,\delta]$. Also by (8.22), ω_∞ has to be Kähler-Einstein. This shows that $M_\infty \backslash \mathcal{S}$ is a smooth manifold on which d_∞ coincides with a Kähler-Einstein metric ω_∞. We will identify d_∞ with ω_∞.

Similarly, we can show that each tangent cone $\mathcal{C}_x$ is regular outside $\mathcal{S}_x$: Let $x \in M_\infty$ and $r_j \to 0$ such that $(M_\infty, r_j^{-2}\omega_\infty, x)$ converge to $(\mathcal{C}_x, \omega_x, o)$. Choose $x_i \in M$ such that $\lim x_i = x$. For each j, we choose $i(j)$ sufficiently large such that $(M, r_j^{-2}\tilde{\omega}_{i(j)}, x_{i(j)})$ converge to $(\mathcal{C}_x, \omega_x, o)$. Furthermore, if $i = i(j)$ is sufficiently large, we have

$$\begin{aligned} &\int_M |\mathrm{Ric}(r_j^{-2}\tilde{\omega}_{i(j)}) \,-\, r_j^{-2}\tilde{\omega}_{i(j)}|\,\left(r_j^{-2}\tilde{\omega}_{i(j)}\right)^n \\ \le\;\; & r_j^{2-2n} \int_M |\mathrm{Ric}(\tilde{\omega}_{i(j)}) - \tilde{\omega}_{i(j)}|\,\tilde{\omega}_{i(j)}^n \,\to\, 0. \end{aligned} \quad (8.23)$$

Clearly, for each j, $r_j^{-2}\omega_i(r_j^2 t)$ is a solution of (8.17) with $\lambda_0 = r_j^{-2}$. Thus, as above, we can use Proposition 8.2 and 8.4 to prove that $\mathcal{C}_x$ is smooth outside $\mathcal{S}_x$ and ω_x is Kähler and Ricci-flat.

Finally, using the smallness of the integral in (8.23) and the slicing argument in [CCT02], one can prove that $\mathcal{S}_{2n-2}$ is empty, so M_∞ is smooth outside a closed subset of complex codimension at least 2. The theorem is proved.

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