My response to CDS' note

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Contents

1 Introduction

This is a response to a recent note by Chen-Donaldson-Sun on

1. Partial C^0-estimate I proposed;

2. My paper "K-stability and Kähler-Einstein metrics".

It is known to be a very difficult problem to derive interior C^0 or C^2-estimates for complex Monge-Ampere equations. Very little is known. These interior estimate may not exist in usual sense. On the other hand, it is known that Calabi's problem on Kähler-Einstein metrics is reduced to solving a complex Monge-Ampere equation. In order to overcome the difficulty due to lack of interior estimates, I proposed the partial C^0-estimate in my solution of Calabi's problem for complex surfaces in late 80s and subsequent publications, for example, in my 1990 ICM lecture as Chen-Donaldson-Sun already noted. It plays a crucial role in solving Calabi's problem on the existence of Kähler-Einstein metrics on Fano manifolds.

In their note, Chen-Donaldson-Sun made a number of unfair accusations against me based on their wishful thinking and false speculations. I will respond to these accusations of theirs. I also note that the second part is related more directly to my submission involved. [1]

[1] For convenience, I will list references as Chen-Donaldson-Sun did in their note.

2 Partial C^0-estimate

In 1994/95, I introduced the notion of K-stability and proved that if the underlying manifold M admits a Kähler-Einstein metric and has no non-zero holomorphic fields, then M is K-stable. The paper was submitted in 1996 and published in 1997. I also realized then that if one can establish the partial C^0-estimate I proposed before, one can prove the converse. Motivated by this, I worked with Cheeger and Colding to prove some compactness theorems for Kähler manifolds, however, those theorems, though very strong for Kähler-Einstein manifolds, are insufficient for proving the partial C^0-estimate in general cases which is needed for old continuity method. Indeed, I was attracted by some other problems after 1998 and worked on this only on an on-and-off bases.

In 2009, I decided to reexamine what I had on the problem of the existence of Kähler-Einstein metrics and rethink about it, so I wrote a 40-pages expository paper

[1] G. Tian: Einstein metrics on Fano manifolds. Progress in Mathematics, volume 239. Birkhäuser, 2012.

This paper was submitted for a proceeding at the end of Feb., 2010. I also sent this to quite a few people,[2] including X.X. Chen on March 4, 2010 and S. Donaldson on April 19, 2010.

In the abstract of this paper, I said

I will describe a program I have been following for the last twenty years. It includes some of my results and speculations which were scattered in my previous publications or mentioned in my lectures.

In July of 2010, S. Donaldson posted a paper

[2] S. Donaldson: Stability, birational transformations and the Kähler-Einstein problem.

In this paper, Donaldson described his program on proving the existence of Kähler-Einstein metrics. To my understanding, one key idea is to introduce a notion of b-stability, the other is to establish a volume estimate based on the compactness theorem of Cheeger-Colding-Tian and use this to prove that the b-stability is equivalent to the existence of Kähler-Einstein metrics. Then, he hoped that the b-stability follows from the K-stability. He never mentioned the partial C^0-estimate, nor mention my program. My paper [1] was cited in a superficial way. I did not believe that his proposed approach is a right one. In fact, I told my belief to some people, including X.X. Chen. This should not be interpreted as any disrespects to Donaldson whom I actually held respects. My belief was based only on mathematical reasoning and my experiences in the

[2] I do not know how to post papers on ArXiv. My joint papers were posted by my collaborators. My paper on Kähler-Einstein metrics and K-stability was posted twice in last Nov. and Feb. by someone else using one of my email accounts.

problem. I believed that my approach by using the partial C^0-estimate is the right one.

In April of 2011, Chen and Donaldson posted

[3] X.X. Chen and S. Donaldson: Volume estimates for Kähler-Einstein metrics and rigidity of complex structures. arXiv:1104.4331.

The paper gives a volume estimate needed in Donaldson's approach in [2]. The method originated from one of my papers in early 1990s. I did not feel that my paper was appropriately cited, so I wrote to Donaldson on April 5, 2011 about it and what I had because of the motivation from the partial C^0-estimate conjecture. Three days later, on April 8, I wrote him again for further comments:

First I want to emphasis that I have no interest in being involved in the volume estimate which plays a role in your project on existence of KE metrics. I will stick by my own approach to the existence problem. I wrote to you only for telling what I knew and protecting what I have done in pursuing my own project.

Because of this incident and motivated by progresses on conic Kähler-Einstein metrics in 2011. I started to spend more time on deriving the partial C^0-estimate for Kähler-Einstein metrics by using the Cheeger-Colding-Tian. I realized in early 2012 that the partial C^0-estimate for Kähler-Einstein manifolds is a local version of Proposition 4.13 in [1] if M_∞ is replaced by Kähler-Ricci flat tangent cones and my old methods can be applied. But I did need to fill in details. At that time, I was writing a proceeding paper for Calabi as invited by Chen and Donaldson. I planned to discuss how to get the partial C^0-estimate at the end of this paper. My writing was slow. Meanwhile, I started to think about extending Cheeger-Colding-Tian to conic Kähler-Einstein metrics because I was aiming at solving the problem on the existence of Kähler-Einstein metrics for K-stable Fano manifolds. Part of my project in extending Cheeger-Colding-Tian was in joint efforts with Z. L. Zhang.

Only in May of 2012, I was told that Donaldson claimed he could do the case for dimension 3. I was not surprised because I had told quite a few people, including X.X. Chen, in many occasions before that if I wanted, I could prove the partial C^0-estimate for Kähler-Einstein metrics and then use the classification of Fano 3-manifolds to solve Calabi's problem in dimension 3. This belief was based on a method of mine used in [3] and Cheeger-Colding-Tian's solution for a conjecture in my ICM 1990 lecture on Kähler-Ricci flat tangent cones in dimension 3. I thought that Donaldson and Sun may have used the same method because of [3]. On the other hand, I was annoyed by that neither Donaldson nor Chen said anything to me that they were turning to the partial C^0-estimate. If they had told me earlier that they were turning to my approach of doing partial C^0-estimate, I would have been happy to communicate with them. So I continued to work on my own and towards the solution of Calabi's problem. By the way, I did not know the existence of the video they mentioned, so I did not and will not see it either.

I finished the first draft of the proceeding paper mentioned above in early June. In the morning of June 13, I sent Donaldson an email with this draft attached and said:

first I apologize for delay in sending you my contribution for the Calabi volume! Hope that it is not too late. I was a bit slow in writing up things. The attached is what I have done. I may do a bit refinements and add a few more paragraphs about questions and speculations if I still have some time. In the attached, you may find some interesting things, especially, about some progresses on partial C^0 estimate based on my old techniques. I thought more since we communicated last year about your paper with Xiuxiong on volume.[3]

I also sent a few others this proceeding paper. I was then attending a summer school at ICTP, Cladio Arezzo, after seeing my paper, told me a few hours later that Donaldson and Sun had just posted a paper which claims the partial C^0-estimate. So I wrote to Donaldson again:

I am at ICTP. Claudio just told me you and Sun posted a paper on partial C^0-estimate. I did not know that. It was a coincidence. I should have communicated with you earlier.

Since June 13, I gave a number of talks on the partial C^0-estimate, two of which are mentioned in the note of Chen-Donaldson-Sun. I always mentioned them and never thought of suggesting I announced earlier as they claimed in the second paragraph of page 3 in their note. As a matter of fact, I did not want to have a fight with them because of priority even if I did the partial C^0-estimate for Kähler-Einstein metrics independently. I wanted to concentrate on the case of conic Kähler-Einstein metrics since my aim was to solve the existence problem on Kähler-Einstein metrics.

On Sept. 19 of 2012, right before I headed for Paris, I wrote to Donaldson

The attached are some notes I wrote. Since I am going to Paris tonight and will talk about them in a conference there, I like to show you since they are related to your recent works. Here are some comments:

1. Section 2 was written in early August. It provides a proof of a theorem claimed in the first draft of my paper for the Calabi proceeding which I sent you in June. You and Sun also gave a proof of this theorem. Originally I did not intend to publicize it, so I sent my proof to only a group of people. But I was told that there are differences between my proof and yours. So it may have some merits. I may just include this proof in the final version of my expository paper for Calabi's proceeding if OK with editors;

2. Section 3 contains a generalization. I intend to use it to prove an existence theorem for Kahler-Einstein metrics under the assumption that the K-energy is bounded. However, there is a technical issue which you can find in the section. It leads to an interesting problem;

[3] I meant the paper [3]

3. In Section 4, I discuss the extension to conic Kahler-Einstein metrics. I think I have a proof. The proof needs new technical inputs. There are two issues one has to take care in such an extension: 1. Extend Cheeger-Colding-Tian to the conic case. This needs some extra arguments because of conic singularity; 2. To construct an almost isometry from some $K^{-\ell}$ to the trivial bundle on a certain tangent cone. One has to be careful here because of codimension 2 singularity which may cause non-trivial honolomy.

The attached notes are what Chen-Donaldson-Sun referred at the end of page 2 in their note. Here is what I said in the introduction and they quoted partly:

Theorem 1.4 and 1.6 were announced with an outlined proof in our expository paper for the proceeding of Calabis 85th birthday edited by Bourguignon, Chen and Donaldson. ...

The expository paper means the one I sent to Donaldson on June 13. As I said, I had no intention to claim that I announced first. Also in the reference, I cited their paper as follow:

[4] Donaldson, S and Sun, S: Gromov-Hausdorff limits of Kähler manifolds and algebraic geometry. arXiv:1206.2609.

This clearly shows my acknowledgement that they presented their proof with details first. Also in my submitted paper on Kähler-Einstein metrics, I cited their paper before mine.

I did not get any response for my email dated on Sept. 19, so I went on to focus on the partial C^0-estimate for conic Kähler-Einstein metrics and started to write up my solution for the existence of Kähler-Einstein metrics on K-stable Fano manifolds.

Now we comment on Chen-Donaldson-Sun's remarks on page 3 about the following paper:

[5] G. Tian: Partial C^0-estimate for Kähler-Einstein metrics. Comm. Mat. Stat. I, 105-113.

Comm. Mat. Stat. is a new journal. When I was attending a meeting at the end of April, I was invited to submit a paper. So I took out the part on partial C^0-estimate for Kähler-Einstein metrics from my previous preprint and added some remarks. I think that it is worth being shown to others. This was after I realized that my proceeding paper may not get published.

The proof is correct though there may be some typos and sloppy writings which are not serious gaps as they claimed. The key reason for Lemma 2.4 to hold is because the cone Kähler metric on tangent cones is of the form $\partial\bar{\partial}\rho$, where ρ is essentially the square of the distance function to the vertex. This is due to Cheeger-Colding-Tian. It is clear that we concern only Kähler-Einstein metrics, then what I claimed is true. Moreover, to get the partial C^0-estimate, we only need a smaller open subset in the regular part which has finite first homology. It is clear from the proof. The proof for this lemma does not need any complicated techniques. Since I do not need to use this paper in my submitted paper, I won't discuss more here.

They complained about my not mentioning their papers on the existence of Kähler-Einstein metrics at the end of page 3. This is because the papers are not needed in [5]. Note that they never mentioned my papers.

3 Stability and Kähler-Einstein metrics

This section is related directly to my submitted paper and concerns my solution on the following conjecture

Conjecture: *If M is a K-stable, then it admits a Kähler-Einstein metric.*

Some literatures mentioned a Yau conjecture. It was not clear at all in his writing which stability should be used, certainly, K-stability was not mentioned.

In my email to Donaldson on Sept 19, I said: *I think that I have a proof* for the partial C^0-estimate for conic Kähler-Einstein metrics. I was not aware of any clear statement from Chen-Donaldson-Sun, or part of this group, on this or **Conjecture** before my talk on Oct. 25, 2012. The partial C^0-estimate is crucial in my solving the above conjecture.

On Oct. 25 of 2012, I announced a solution for **Conjecture** and outlined its proof.

[6] G. Tian: Conic Kähler-Einstein metrics.[4]

I made clear that we use the deformation method proposed by Donaldson in 2010 and modified by Li-Sun later through conic Kähler-Einstein metrics. The modification is simple but crucial. Deforming method through conic metrics was used in the gauge theory in 80/90s and in the study of 3-manifolds. I have to point out that the continuity method was proposed in Donaldson's attempt to solve **Conjecture** by using the b-stability as I mentioned in previous section. The proposed method becomes useful (at least for our current solution) only when it is combined with my approach in which the partial C^0-estimate is the core. It is a common sense that the most difficult part in any continuity method is to establish a prior estimates. The partial C^0-estimate plays the role in this situation. So the final approach to solving the above conjecture is a combination of the continuity method originated in Donaldson's b-stability approach and the partial C^0-estimate from my approach of using old continuity method.

Their statement "*Tian's lecture gave few details, and proofs of some of the key assertions made have never appeared"* is not true. I gave many details in my one-hour lecture. I discussed how to approximate conic Kähler-Einstein metrics by smooth Kähler metrics with precise lower bound on Ricci curvature. This is very important in my proof and enables us to extend some of Cheeger-Colding and Cheeger-Colding-Tian. This approximation theorem was not known before.

[4] http://www.math.sunysb.edu/Videos/Cycles2012

I showed key steps of proving this. I said that this approximation theorem is important in using my work with B. Wang to deal with the case that the cone angle tends to 1. I also showed the main idea how to show the smooth convergence in the presence of cone singularities. This is new and crucial in extending Cheeger-Colding-Tian compactness theorem to conic Kähler-Einstein metrics. This extended compactness theorem is crucial in establishing the partial C^0-estimate.

Two incidents are worth being noted. 1. At the end of my talk, C. Lebrun asked one question concerning Chen and Donaldson's (Sun was not mentioned) work on the conjecture, I replied that they had not told me anything about it. Indeed, I had not been aware of any statement from either Chen or Donaldson on a proof of the conjecture. 2. On the next morning after my talk, Chen came to shake hands with me when I was chatting with R. Schoen before a talk, he did not say anything about their proof. Indeed, he was not in presence of my talk, but I suspect that he was informed in time.

On Nov. 20, 2012, with someone's help, I posted the first version of my paper which contains a proof of my theorem

[7] G. Tian: K-stability and Kähler-Einstein metrics. arxiv.org/pdf/1211.4669

On Jan. 28 of 2013, with someone's help, I uploaded a newer version with some typos corrected and one appendix added. This appendix fills in a proof of a lemma I stated and explained in my previous version. The proof of this lemma is based on an application of partial C^0-estimate and a well-known fact in geometric analysis. On Feb. 17 of 2013, I submitted my paper with one more appendix added for the readers' convenience. This appendix is an outline of my previous work with B. Wang. I also sent this submitted version to quite a few people around that time.

As they noted, on Oct. 28 of 2012, Chen-Donaldson-Sun rushed to post an announcement. Subsequently, they posted three papers to claim a proof for the theorem I first claimed and proved. Their last paper appeared in Feb. 1 of 2013, so I have the priority in both announcing the theorem and providing a detailed proof.

While I always mentioned them in my writings and talks, I was told that they never mentioned my work in their talks or papers. Apparently, they were aware of my talk on Oct. 25 and my paper because some of them already attacked me in various ways. I do not think that they acted professionally and their practice is good for our mathematical community.

Before I go to discuss technical issues, I would also like to note that their objections at the end of page 5 are based on their wishful thinking and speculations. From what I quoted and discussed above, I first announced the solution for **Conjecture** and provided the first detailed proof. Also my technical response below will show that their objections are not valid.

In the following, I will address the technical issues they raised.

3.1 Reductiveness

This subsection concerns Section 6 in my paper and corresponding section of Chen-Donaldson-Sun's note. The issue is how to get the existence of Kähler-Einstein metrics from the K-stability assuming we have proved the partial C^0-estimate. This issue is identical for the old continuity method and the new one through conic metrics. In Section 6 of my paper, I gave a more direct proof and outlined an alternative way of completing the proof. Both follow what I knew when I used the old continuity method long ago.

First make precisely the issue at hand: We embed (M, D) into some projective space $\mathbb{C}P^N$ by bases of $H^0(M, K_M^{-\ell})$. All these embeddings differ by transformations in $G = \mathbf{SL}(N+1, \mathbb{C})$ which acts on $\mathbb{C}P^N$. We assume that M_∞ is a normal variety with a divisor D_∞ and (M_∞, D_∞) is in the closure of the orbit of (M, D) under G-action. We need to prove that there is a $\mathbb{C}^*$-subgroup $G_0 \subset G$ such that (M_∞, D_∞) lies in the closure of $G_0[(M, D)]$. One does not need to worry about this issue if one defines the stability in terms of group action instead of $\mathbb{C}^*$-actions.

It is a known fact in algebraic geometry that such a G_0 above exists if the stabilizer of (M_∞, D_∞) in G is reductive. It was known to me, as well as others who have good knowledge in the Geometric Invariant Theory. It was also shown in Donaldson's 2010 paper "Stability, birational transformations and the Kahler-Einstein problem " where he presented his unsuccessful approach by using the b-stability. The preprint was given to me in 2010. It was a fact at least by 2012. So it is absurd for them to use this against me by saying *The argument for this does not seem to be widely known, and this observation is an important ingredient in our work. It features in the announcement.*

My first proof is to show that the stabilizer of (M_∞, D_∞) in G is reductive, i.e., Lemma 6.3. It is an extension of Matsushima's proof to normal varieties with a suitable Kähler-Einstein metric. Here are their objections: The first one is about a formula of θ_∞ (cf. page 7 of their note), clearly, this is a typo and X is missing. As I corrected in a subsequent version, the right formula is

$$\theta_\infty = \theta + \frac{1}{\ell} X(\rho_{\omega_\infty, \ell}).$$

This formula was in a number of my previous publications.

Their second objection is about Lipschtz continuity of θ_∞. It follows from the Moser iterations and is indeed standard. I included more details to make it more readable in my submitted version and added a few more lines recently upon the request by the editorial board. To be helpful, I include a summary here:

Let ϕ_t be an one-parameter subgroup of automorphisms generated by the real or imaginary part of X, then

$$\phi_t^* \omega_\infty \, = \, \frac{1}{\ell}\, \omega_{FS} \, + \, \sqrt{-1}\, \partial\bar\partial\, \psi_t.$$

Since ω_∞ is Kähler-Einstein, we have

$$(\omega_{FS}|_{M_\infty} \, + \, \sqrt{-1}\partial\bar\partial\psi_t)^n \, = \, ||\sigma_\infty||_0^{-2(1-\bar\beta)}\, e^{-\bar\mu\, \psi_t}\, \Omega,$$

where Ω is a volume form on M_∞ corresponding to a Hermitian metric with curvature ω_{FS} and $||\cdot||_0$ is a given Hermitian metric.

Differentiating this equation on t at 0 for both ϕ_t induced by the real and imaginary part of X, we see that in a weak sense,

$$\Delta_\infty \theta_\infty + \bar{\mu}\,\theta_\infty = 0,$$

where Δ_∞ denotes the Laplacian of ω_∞.

The partial C^0-estimate implies that θ_∞ belongs to L^p for some $p > 1$. Since we have the Sobolev inequality and Poincare inequality for $(M_\infty, \omega_\infty)$, it follows from the Moser iteration that θ_∞ is bounded.

Next we have the following Bochner identity:

$$\frac{1}{2}\Delta_\infty |\nabla\theta_\infty|^2 = |\nabla\bar{\partial}\theta_\infty|^2 - \bar{\mu}|\nabla\theta_\infty|^2.$$

Then a Moser iteration shows that θ_∞ is Lipschtz continuous.

One can see more details in the revised version of my paper sent to the journal last August. I think that I had sufficient details, but I will be happy to write more in my paper upon requests from the referees.

My second proof is related to the CM stability I introduced in 90s and S. Paul and I studied in early 2000. The CM-stability is also an algebraic condition and imitates the definition of the Chow-Mumford stability in terms of properties of orbits under G-action.

In my paper, by using the partial C^0-estimate, I proved the following theorem (Theorem 6.6):

Theorem 3.1. *Let M be a Fano manifold without non-trivial holomorphic fields, then M admits a Kähler-Einstein metric if and only if M is CM-stable.*

This is of independent interest and deserves to be published in a leading journal. If I had titled my paper as "Stability and Kähler-Einstein metrics", they could have had no objection on reductiveness. I feel that they may lack the insight of the whole picture on stability. I started the study of K-stability and CM-stability. The CM stability should imply the K-stability. This is an algebraic problem and was a project of my former student, Sean Paul. Since 2011 or earlier, Sean has proposed studying stability for pairs which extends the classical Geometric Invariant Theory. Let us mention the work of Sean in 2008 and which I cited and was published in Annals

[8] S. Paul: Hyperdiscriminant polytopes, Chow polytopes, and Mabuchi energy asymptotics. Ann. of Math. (2) 175 (2012)

It follows from this work that the K-stability or CM-stability are reduced to defining stability of pairs in terms of $\mathbb{C}^*$-actions or property of orbits. Then the problem is whether or not these two definitions are equivalent in Sean's formulation.of stable pairs. In the August of 2012, Sean told me that he can do the case for semi-stable pairs and posted his preprint on ArXiv on Oct. 2. See

[9] S. Paul: A Numerical Criterion for K-Energy maps of Algebraic Manifolds. arXiv:1210.0924.

I think that Sean's approach is conceptually better. I studied this paper and knew that his arguments can be modified to cover the stable case, but I preferred to letting Sean finish what he started. Indeed, he has posted a paper for the stable case:

[10] S. Paul: Stable Pairs and Coercive Estimates for The Mabuchi Functional. arXiv:1308.4377.

Since only the above two proofs were presented in my submitted paper, there is no need to say much about their accusation on my talk on July 2013. The accusation is ridiculous. However, I feel that Berndtsson and Berman-Boucksom-Essydieux-Guedj-Zeriahi deserve more credits for reductiveness. On Nov. 2 of 2012, Berman-Boucksom-Essydieux-Guedj-Zeriahi posed a revised version of their 2011 preprint:

[11] Kähler-Einstein metrics and the Kähler-Ricci flow on log Fano varieties. arXiv:1111.7158,

In this paper, using some ideas of Berndtsson, they proved a stronger uniqueness theorem for Kähler-Einstein metrics on normal varieties. More precisely, they showed that given two Kähler-Einstein metrics ω_0 and ω_1, there is a family of automorphisms ϕ_t $(0 \leq t \leq 1)$ such that $\omega_1 = \phi_1^*(\omega_0)$ and $\phi_t(\omega_0)$ form a (weakly) geodesic in the space of Kähler metrics. This immediately implies that Kähler-Einstein metrics are unique modulo a reductive subgroup of automorphisms and consequently, the reductiveness of the automorphism group. So I feel that Berndtsson and Berman-Boucksom-Essydieux-Guedj-Zeriahi had essentially proved the reductiveness and deserve more credits.

I was not aware of the paper of Berman-Boucksom-Essydieux-Guedj-Zeriahi until early this year. I may have been too concentrating on my own approaches. I apologize to them for not citing them and giving them sufficient credits in my paper before.

3.2 The case when the cone angle is less than 2π

This concerns Appendix 1 in my paper. In this appendix, I gave a detailed proof of Lemma 5.8. My competitors claimed that I did not know the proof. It is false and based on their wishful thinking.

The issue is to construct a cut-off function which supports in the regular part of a tangent cone $\mathcal{C}_x$ (see Lemma 5.8 in my paper).

If the singular set is a subvariety, such a cut-off function can be constructed easily. This had been known to me when I was a student many years ago. The basic reason is because the Poincare metric on punctured disc has finite volume as I mentioned in the first version of my paper.

In our case, though the singular set of $\mathcal{C}_x$ may not be a priori a subvariety, we can use the partial C^0-estimate to reduce to the situation of a subvariety modulo a subset of higher codimension. More precisely, we use the partial C^0-estimate inductively and get required information near those singular points where some tangent cones are simple and their singular sets are subvarieties. The key is again the partial C^0-estimate I proposed and studied for long. An additional ingredient is the slicing argument which I used in my joint work with Cheeger-Colding and work with T. Riviere on regularity of pseudo-holomorphic currents. Though I showed several ways of proving Lemma 5.8 which are all based on the partial C^0-estimate, I chose one which I think the simplest and gave all the necessary details. I did not, and do not, know how they did.

Let me list two more facts:

1. In my email to S. Donaldson dated on Sept. 19 of 2012 and mentioned above, I wrote

I discuss the extension to conic Kahler-Einstein metrics. I think I have a proof. The proof needs new technical inputs. There are two issues one has to take care in such an extension: 1. Extend Cheeger-Colding-Tian to the conic case. This needs some extra arguments because of conic singularity; 2. To construct an almost isometry from some $K^{-\ell}$ to the trivial bundle on a certain tangent cone. One has to be careful here because of **codimension 2 singularity** *which may cause non-trivial honolomy.*

This shows that I paid attentions to possible technical issues which may arise from singular set of codimension 2.

2. In my email to S. Donaldson dated on June 13 of 2012, I attached the first draft of my paper for Calabi's proceeding. On page 19, I had a paragraph:

Thus by replacing ℓ by $k\ell$ for a large k, we get an embedding $\Phi_i : M_i \mapsto \mathbb{C}P^N$ by using an orthonormal basis $\{\sigma_a^i\}$ of $H^0(M_i, K_{M_i}^{-\ell})$ such that Φ_i converge to a rational map $\Phi_\infty : M_\infty \dashrightarrow \mathbb{C}P^N$ which is a holomorphic embedding near x. It follows that M_∞ is a variety near x and the singular stratum S_2 is a subvariety near x. Since x is arbitrary in S_2, $M_\infty \backslash \bigcup_{m>2} S_m$ is a variety with singular set S_2.

This is similar to the situation in proving Lemma 5.8. When tangent cones are simple, we can apply the partial C^0-estimate to show the singular set is locally a subvariety. In Appendix 1, we apply the similar idea to singularity of codimension 2 instead of 4.

These two facts show that I was aware of the problem of codimension 2 singularity and the method how to deal with it.

Also, as acknowledged in Donaldson and Sun's paper [4], it was me who first pointed out the use of the partial C^0-estimate to prove that the singular set is subvariety. The proof of Lemma 5.8 follows this line of ideas.

Their second objection is about Lemma 5.5. First, it is correct and $\bar{\mathcal{S}}_x$ is closed. The closedness follows directly from the statement in the paragraph before Lemma A.1. in my Appendix 1. Secondly, this closedness is not used in proving Lemma 5.8 and subsequent proof of the partial C^0-estimate, so we do not need to worry their objection.

3.3 The case when the cone angle tends to 2π

There is no need to say much about this objection of them. What we need for convergence to tangent cones follows easily from Theorem 4.3, see the Remark after Theorem 4.3 in my submitted version. It is straightforward.

By the way, the proof of Theorem 4.3 is definitely correct and was the last technical problem I solved during the course of my solving the **Conjecture**. Also I discussed the main steps of its proof in my lecture on Oct. 25 of 2012.

There is no problem from Theorem 4.3 to partial C^0-estimate. The arguments in Section 5 apply. Note that D_∞ is a divisor modulo a subset of higher codimension. One can also run the L^2-estimate directly on M_∞.

3.4 One more remark

In their note, Chen-Donaldson-Sun mentioned the paper of Jeffres-Mazzeo-Rubinstein. Indeed, I quoted the paper in mine. Their paper may have a problem in deriving the C^3-estimate. But they have provided an alternative proof for that. Also I did think about the C^3-estimate in the paper of Jeffres-Mazzeo-Rubinstein. I think the problem is fixable and the C^3-estimate can be obtained by existing techniques. Nevertheless, Chen-Donaldson-Sun already claimed a correct proof within a couple of months, so it is a fixable problem even according to them.